# Sur le problème de Cauchy pour des EDP quasi-linéaires dispersives

**Thèse de Doctorat de l'Université de Cergy-Pontoise**

présentée par Tristan Robert

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# Résumé

Dans cette thèse, on s'intéresse au problème de Cauchy pour des équations quasi-linéaires dispersives. Pour une telle équation, l'enjeu est de montrer l'existence et l'unicité d'une solution de l'équation avec une donnée initiale prescrite dans un espace fonctionnel le plus large possible. Nous étudierons deux modèles décrivant l'évolution de la surface d'un fluide satisfaisant certaines conditions physiques.

La première partie est consacrée à l'étude de l'équation de Kadomtsev-Petviashvili avec forte tension de surface (KP-I). Cette équation possède une structure Hamiltonienne et admet donc une fonctionnelle d'énergie préservée par le flot. Afin d'obtenir des solutions définies globalement en temps, on cherche donc à construire un flot dans l'espace de Banach naturellement associé à cette énergie. De plus, on se restreint à des espaces contenant des solutions particulières (les solitons linéaires de KdV), on impose donc une condition de périodicité dans la direction transverse à la propagation du fluide.

On commence par illustrer le caractère quasi-linéaire de l'équation en montrant a priori que le flot dans cet espace ne peut pas être très régulier. Ceci restreint l'éventail des méthodes connues pour résoudre ce type de problème. On a donc recours à la méthode dite de restriction de la transformée de Fourier en temps petits développée récemment par Ionescu, Kenig et Tataru pour traiter ce même modèle sans condition de périodicité. On obtient ainsi l'existence globale et l'unicité de la solution du problème de Cauchy dans l'espace d'énergie. Enfin, on montre que le flot ainsi construit est continu mais pas uniformément continu sur les ensembles bornés de l'espace d'énergie.

Une application intéressante de la construction d'un flot global sur l'espace d'énergie contenant les solitons linéaires est de lever une restriction sur les perturbations admissibles dans un résultat de Rousset-Tzvetkov sur la stabilité orbitale des solitons linéaires de faible vitesse.

Dans la deuxième partie de la thèse, on s'intéresse à l'équation KP-I d'ordre cinq, qui est une alternative au modèle précédent dans le cas d'une tension de surface avoisinant une valeur critique pour laquelle l'effet dispersif devient plus faible. Pour cette équation, le comportement quasi-linéaire ne se manifeste que pour des données périodiques dans la direction transverse, et les autres cas avaient été étudiés précédemment dans les travaux de Saut et Tzvetkov. On considère ici des données également périodiques dans la direction de propagation. On montre que pour certains choix de périodes, le flot ne peut pas être régulier. Afin de traiter le problème indifféremment des périodes spatiales, on utilise donc une nouvelle fois la méthode précédente pour construire un flot global dans l'espace associé au Hamiltonien de ce modèle.



# Abstract

This thesis investigates the Cauchy problem for some quasilinear dispersive equations. Being given such an equation, the goal is then to construct a unique solution to this equation with a prescribed initial data belonging in a function space as large as possible. We will study two models describing the time evolution of the surface of a fluid in a particular regime.

The first part of this thesis is devoted to the study of the Kadomtsev-Petviashvili equation in the case of strong surface tension (KP-I). This equation has a Hamiltonian structure, so it admits an energy functional which is preserved under the flow. In order to recover solutions which are globally defined in time, we thus seek to construct a flow map in the Banach space naturally associated with the energy. In addition, we restrict ourself to spaces including some special solutions (the KdV line soliton), so we require the functions to be periodic in the transverse direction.

We start by illustrating the quasilinear behaviour of the equation : we show that a flow map defined on this space cannot be too regular. This limits the range of applicable methods known to solve this kind of problem. We thus use the so-called small times Fourier restriction norm method recently developped by Ionescu, Kenig and Tataru to deal with the same model without the periodicity assumption. We thereby obtain the global existence and uniqueness of a solution to the Cauchy problem in the energy space. At last, we prove that the flow map constructed this way is continuous yet not uniformly continuous on the bounded sets of the energy space.

An interesting application of the construction of a global flow on the energy space containing the line solitons is to get rid of an extra condition on admissible perturbations in a result of Rousset-Tzvetkov on the orbital stability of the small speed line solitons.

In the second part of the thesis, we turn to the fifth-order KP-I equation, which is an alternative to the previous model should the tension surface come close to a critical value in which the dispersive effect becomes weaker. Regarding this equation, the quasilinear behaviour only manifests when solutions are periodic in the transverse direction, and the other cases were treated in the work of Saut and Tzvetkov. We study the case of functions which are also periodic in the direction of propagation, and we show that at least for some choice of periods the flow map fails to be smooth. In order to treat the problem regardless of the periods, we make another use of the method above to construct a global flow in the space associated to the Hamiltonian of the equation.

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# Chapitre 0

# Introduction

Ce chapitre est une version française abrégée des chapitres 1 et 2.

## 0.1 Généralités

Le but de cette thèse est de construire des solutions pour des équations aux dérivées partielles dispersives non-linéaires. De telles équations se mettent sous la forme d'un problème d'évolution

$$\partial_t u = \mathcal{L}u + \mathcal{N}(u), \tag{0.1.1}$$

où la fonction inconnue $u : (t, \mathbf{x}) \in \mathbb{R} \times \Omega \mapsto u(t, \mathbf{x}) \in \mathbb{R}$ appartient à un espace fonctionnel $X$ en tout temps, $\Omega$ est un domaine dans $\mathbb{R}^d$ (avec conditions au bord périodiques en présence d'un bord), $\mathcal{L}$ est un opérateur linéaire anti-adjoint sur $X$ et $\mathcal{N}$ est une non-linéarité. De tels modèles d'évolution se rencontrent fréquemment dans l'étude de phénomènes physiques ondulatoires et non-linéaires, en particulier comme modèles de dynamique des fluides dans certains régimes. Dans cette thèse, on considère des modèles de type KP, dans lesquels $d = 2$, $\Omega = \mathbb{R} \times \mathbb{T}$ ou $\Omega = \mathbb{T}^2$ (et $\mathbb{T}^d$ est le tore de dimension $d$) et les opérateurs ont la forme particulière

$$\mathcal{L} = \partial_x A \tag{0.1.2}$$

et

$$\mathcal{N}(u) = \partial_x f(u), \tag{0.1.3}$$

avec $A$ un opérateur (pseudo) différentiel symétrique et $f : u \in \mathbb{R} \mapsto u^p$ où $p \in \mathbb{N}$ avec $p > 1$.

Les équations de Kadomtsev-Petviashvili ont été introduites en 1970 [KP70] par les deux auteurs afin de modéliser l'évolution vers la droite sur l'axe des abscisses de la surface d'un fluide incompressible, irrotationnel et non visqueux de faible profondeur sur un fond plat, présentant des oscillations de faible amplitude et grande longueur d'onde avec une faible dépendance dans la direction transverse à la propagation.

L'équation KP-I

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{0.1.4}$$

correspond à une forte tension de surface, alors que l'équation KP-II

$$\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{0.1.5}$$

modélise le fluide avec une faible tension de surface. En fait, la tension de surface intervient devant le terme de dispersion $\partial_x^3$ dans le modèle. En particulier, pour une valeur critique des paramètres physiques, le coefficient devant ce terme s'annule. Il faut alors considérer un régime moins non-linéaire, modélisé cette-fois par l'équation KP-I d'ordre 5

$$\partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0. \tag{0.1.6}$$

Les opérateurs $\mathcal{L}$ et $\mathcal{N}$ considérés dans cette thèse ont la forme particulière (0.1.2)-(0.1.3). Ainsi, l'équation (0.1.1) peut-être vue comme une équation Hamiltonienne

$$\partial_t u = J \cdot \nabla\mathcal{H}(u), \tag{0.1.7}$$

où l'opérateur symplectique est ici $J = \partial_x$ et le Hamiltonien est donné par

$$\mathcal{H}(u) = \frac{1}{2}\langle Au, u\rangle + \frac{1}{(p+1)}\int_\Omega u(x)^{p+1}\mathrm{d}x, \tag{0.1.8}$$

où $\langle\cdot\rangle$ désigne le produit scalaire dans $L^2(\Omega)$.

Pour une équation de la forme (0.1.7), l'énergie $\mathcal{H}$ est (au moins formellement) préservée par le flot. De plus, en multipliant l'équation par $u$ et en intégrant en $x$, puisque $\mathcal{L}$ est anti-adjoint et $f$ est réelle, on obtient que la norme $L^2(\Omega)$ de $u$ est également conservée par le flot.

En pratique, cela signifie que si $X = L^2(\Omega)$, ou si $X$ est un espace dont la norme est plus faible que $\langle A\cdot, \cdot\rangle$, et si $u$ est une solution définie sur $[0;T]$ où $T > 0$ dépend de la taille de la donnée initiale, alors les quantités conservées fournissent un contrôle a priori sur la norme de la solution, qui permet d'étendre celle-ci à un intervalle de temps quelconque.

Une fois obtenu un modèle du type (0.1.1), la problématique est la suivante : étant donné $u_0$ dans un espace fonctionnel $X$, on cherche une unique solution au problème

$$\begin{cases} \partial_t u = \mathcal{L}u + \mathcal{N}(u), \\ u(t=0) = u_0 \in X. \end{cases} \tag{0.1.9}$$

Plus précisément (cf. [Tzv04])

**Définition 0.1.1**

*Le problème de Cauchy (0.1.9) est dit* (localement) bien posé *dans $X$ si les conditions suivantes sont satisfaites :*

*(i) pour tout ensemble borné $B \subset X$, il existe un temps $T > 0$ et un espace de Banach $\mathbf{X}_T$ qui s'injecte continûment dans $\mathcal{C}([0;T], X)$ tels que pour toute donnée initiale $u_0 \in B$ il existe une unique solution $u \in \mathbf{X}_T$ définie sur $[0;T]$ et vérifiant (0.1.1) sur $[0;T]$*

*(ii) le flot $\Phi : u_0 \in X \mapsto u \in \mathbf{X}_T$ ainsi défini est continu*

**Remarque 0.1.2.** *Le problème est dit* globalement *bien posé lorsque* $(i)$-$(ii)$ *ont lieu pour tout temps $T > 0$.*

Ainsi, dans le cas où la norme de $X$ est contrôlée par une loi de conservation qui est bien définie, il y a équivalence entre localement et globalement bien posé.

Une stratégie générale pour prouver le caractère bien posé est, en s'inspirant de la preuve du théorème de Cauchy-Lipschitz pour les équations en dimension finie, d'appliquer un argument de point fixe sur une formulation intégrale de l'équation. Dans notre cas, il s'agit de la *formule de Duhamel*

$$u(t) = \mathrm{e}^{t\mathcal{L}}u_0 + \int_0^t \mathrm{e}^{(t-t')\mathcal{L}}\mathcal{N}(u(t'))\mathrm{d}t'. \tag{0.1.10}$$

Lorsque cette stratégie aboutit, la propriété de contraction montre que le flot ainsi obtenu est au moins Lipschitz. On peut alors définir une notion plus fine de caractère bien posé pour discriminer les problèmes selon la régularité du flot (voir [Tzv04] pour une présentation détaillée sur ce point) :

**Définition 0.1.3**

*Le problème (0.1.9) est dit* semi-linéairement bien posé *s'il est bien posé et que le flot* $\Phi : u_0 \in X \mapsto u \in \mathbf{X}_T$ *est* uniformément continu *sur les boules fermées de* $X$*. Sinon, il est dit* quasi-linéaire.

Dans cette thèse, on étudie des cas où le problème (0.1.9) est quasi-linéaire. Dans la section suivante, on va présenter les méthodes de construction de flots pour des équations d'évolution de type (0.1.1).

## 0.2 Concernant le problème de Cauchy pour les EDP dispersives non-linéaires

L'étude des EDP dispersives non-linéaires fait l'objet de travaux de recherches depuis plusieurs décennies, et plusieurs méthodes ont été développées pour construire des flots sur des espaces fonctionnels de plus en plus larges.

La stratégie générale pour obtenir le caractère semi-linéairement bien posé est de trouver des espaces fonctionnels $\mathbf{F}(T) \subset \mathcal{C}([0;T], H^s)$ et $\mathbf{N}(T)$ tels que l'on ait une *estimation linéaire*

$$||u||_{\mathbf{F}(T)} \lesssim ||u_0||_{H^s} + ||(\partial_t - \mathcal{L})u||_{\mathbf{N}(T)}, \tag{0.2.1}$$

et une *estimation non-linéaire*

$$||\mathcal{N}(u) - \mathcal{N}(v)||_{\mathbf{N}(T)} \lesssim ||u-v||_{\mathbf{F}(T)} \left(||u||_{\mathbf{F}(T)} + ||v||_{\mathbf{F}(T)}\right)^{p-1}. \tag{0.2.2}$$

(0.2.1)-(0.2.2) permettent alors de boucler l'estimation de point fixe dans $\mathbf{F}(T)$ (la contraction étant assurée soit par une hypothèse de petitesse sur la donnée soit par le gain d'un facteur $T^{0+}$ dans l'une des deux estimations précédentes).

Le cas le plus simple est de regarder le problème (0.1.9) lorsque $\mathcal{N} = f$, i.e en l'absence de dérivée dans la non-linéarité. Dans ce cas, en notant

$$\mathfrak{D} : u \mapsto \int_0^t \mathrm{e}^{(t-t')\mathcal{L}} u(t')\mathrm{d}t' \tag{0.2.3}$$

l'opérateur de Duhamel, une estimation brutale montre que cet opérateur est continu de $\mathbf{X}_T = \mathcal{C}([0;T], H^s)$ dans lui-même. De plus, l'estimation

$$\begin{aligned} ||f(u) - f(v)||_{H^s} \lesssim C\left(p, ||u||_{L^\infty}, ||v||_{L^\infty}\right) \{ &||u-v||_{H^s} \left(||u||_{L^\infty} + ||v||_{L^\infty}\right) \\ &+ ||u-v||_{L^\infty} \left(||u||_{H^s} + ||v||_{H^s}\right)\}, \end{aligned} \tag{0.2.4}$$

valable pour $s \geqslant 0$ et $u, v \in H^s \cap L^\infty$, montre que $f$ est également bornée de $\mathbf{X}_T$ dans $\mathbf{X}_T$ lorsque $s > d/2$ (par l'injection de Sobolev $H^{(d/2)+} \hookrightarrow L^\infty$). Cela permet d'obtenir (0.2.1)-(0.2.2) avec $\mathbf{F}(T) = \mathbf{N}(T) = \mathcal{C}([0;T], H^s)$.

Cependant, cet argument n'est plus valable lorsque $f$ est remplacée par $\partial_x \circ f$. Une manière de contourner cette difficulté est de regarder un problème approché

$$\begin{cases} \partial_t u^\varepsilon = \mathcal{L}^\varepsilon u^\varepsilon + \mathcal{N}^\varepsilon(u^\varepsilon), \\ u^\varepsilon(t=0) = u_0^\varepsilon \end{cases}, \tag{0.2.5}$$

pour lequel il est plus facile de construire une solution $u^\varepsilon$. On cherche ensuite à obtenir la compacité de la famille de solutions $(u^\varepsilon)$ et, en passant à la limite $\varepsilon \to 0$ à récupérer une solution du problème original (0.1.9).

Généralement, on prend $\mathcal{L}^\varepsilon = \mathcal{L} + \varepsilon\Delta$ afin de bénéficier de l'effet de lissage du noyau de la chaleur pour récupérer la dérivée, et l'existence d'une unique solution $u^\varepsilon$ est garantie par un argument classique. De plus, on a l'estimation d'énergie

$$\frac{\mathrm{d}}{\mathrm{d}t} \left|\left|u^\varepsilon(t)\right|\right|_{H^s}^2 \lesssim \langle \langle D\rangle^s u^\varepsilon(t), \langle D\rangle^s \mathcal{N}^\varepsilon(u^\varepsilon(t))\rangle_s \lesssim \left|\left|\nabla u^\varepsilon(t)\right|\right|_{L^\infty} \left|\left|u^\varepsilon(t)\right|\right|_{H^s}^2 \tag{0.2.6}$$

où $D = -i\nabla$ et la dernière inégalité est obtenue par intégration par parties et par une estimation de commutateur due à Kato et Ponce :

**Lemme 0.2.1 (*[KP88]*)**

*Soient $f \in H^s \cap W^{1,\infty}$ et $g \in H^{s-1} \cap L^\infty$, $s \geqslant 0$, alors*

$$\left|\left|[\langle D\rangle^s, f]\, g\right|\right|_{L^2} \lesssim \left|\left|f\right|\right|_{H^s} \left|\left|g\right|\right|_{L^\infty} + \left|\left|\nabla f\right|\right|_{L^\infty} \left|\left|g\right|\right|_{H^{s-1}}. \tag{0.2.7}$$

En utilisant le lemme de Gronwall et l'injection de Sobolev $H^s \hookrightarrow W^{1,\infty}$ pour $s > 1 + d/2$, on obtient que pour $T$ petit (uniforme en $\varepsilon$) la famille $(u^\varepsilon)$ est bornée dans $\mathcal{C}([0;T], H^s)$ et un argument de compacité permet d'en extraire une sous-suite convergeant vers une solution $u$ unique dans $\mathcal{C}([0;T], H^s) \cap \left\{\partial_x u \in L^1_T L^\infty\right\}$. En utilisant la méthode de Bona et Smith [BS75], on montre enfin que le flot est continu. Ainsi, on arrive au résultat général

**Théorème 0.2.2**

*Le problème de Cauchy (0.1.9) est localement bien posé dans $H^s$ pour $s > 1 + d/2$.*

Le théorème général ci-dessus permet ainsi de construire un flot local dans des espaces $X$ assez restreints : en effet, si la dimension est grande ou l'ordre de $A$ est faible, alors l'espace d'énergie $X$ est strictement contenu dans $H^{(1+d/2)+}$ et on ne peut donc pas en déduire directement que le problème est globalement bien posé.

Par contre, cette méthode n'exploite pas du tout le caractère dispersif de $\mathcal{L}$ : en effet, pour un tel opérateur $\mathcal{L}$, le flot linéaire a la propriété remarquable de "disperser" la donnée initiale. Quantitativement, cela se traduit par des estimations de la forme

$$\left|\left|\mathrm{e}^{t\mathcal{L}} u_0\right|\right|_{L^\infty(\Omega)} \lesssim |t|^{-\alpha} \left|\left|u_0\right|\right|_{L^1(\Omega)} \tag{0.2.8}$$

Avec un argument supplémentaire, on obtient l'estimation de Strichartz : pour certains couples $(q,r)$ le flot linéaire n'est pas seulement unitaire de $H^s$ dans $L^\infty([0;T], H^s)$ mais aussi borné de $L^2$ dans $L^q([0;T], L^r)$, et

$$\left|\left|\mathrm{e}^{t\mathcal{L}} u_0\right|\right|_{L^q_T L^r} \lesssim \left|\left|u_0\right|\right|_{L^2} \tag{0.2.9}$$

Ce gain d'intégrabilité permet, dans le cas où $\mathcal{N} = f$ ne comporte pas de dérivée, de mettre en place directement (0.2.1)-(0.2.2) avec $\mathbf{F}(T) = \mathcal{C}([0;T], H^s) \cap L^q_T L^r$ et $\mathbf{N}(T) = L^{q'}_T L^{r'}$ pour un $s \leqslant d/2$ et un couple admissible $(q,r)$ bien choisi. Cela permet par exemple de montrer que le problème de Cauchy pour l'équation de Schrödinger non-linéaire cubique est globalement bien posé dans $L^2(\mathbb{R})$ [Tsu87].

Cette méthode s'adapte également dans le cas d'une non-linéarité avec une dérivée, mais il faut alors obtenir d'autres estimations dans diverses normes $L^p_t L^q_x$ ou $L^q_x L^r_t$ et avec un gain de régularité. C'est par exemple le cas de l'équation KdV, pour laquelle les estimations de Strichartz

$$\left|\left|\mathrm{e}^{-t\partial_x^3} |D|^{1/4} u_0\right|\right|_{L^4_t L^\infty} \lesssim \left|\left|u_0\right|\right|_{L^2} \tag{0.2.10}$$

de lissage local

$$\left|\left|\partial_x \mathrm{e}^{-t\partial_x^3} u_0\right|\right|_{L_x^\infty L_t^2} \lesssim ||u_0||_{L^2} \tag{0.2.11}$$

et de fonction maximale

$$\left|\left|\mathrm{e}^{-t\partial_x^3} u_0\right|\right|_{L_x^2 L_T^\infty} \lesssim ||u_0||_{H^s} \tag{0.2.12}$$

(pour $s > 3/4$ et $T \in ]0;1]$) permettent d'obtenir le caractère semi-linéairement bien posé dans $H^s(\mathbb{R})$, $s > 3/4$ [KPV91].

Pour obtenir une propriété de contraction dans des espaces moins réguliers et en présence d'une dérivée dans $\mathcal{N}$ ou lorsque la donnée est périodique, il faut alors chercher d'autres espaces $\mathbf{F}(T)$ et $\mathbf{N}(T)$, plus adaptés à l'équation considérée.

Il semble qu'un choix optimal de tels espaces soit réalisé par les espaces $\mathbf{F} = X^{s,b}$ et $\mathbf{N} = X^{s,b-1}$ introduits par Bourgain [Bou93a] pour traiter les équations de Schrödinger non-linéaire et Korteweg-de Vries périodiques (voir aussi [RR82] dans le cas des ondes non-linéaires). Ces espaces sont définis comme l'adhérence de la classe de Schwartz pour la norme

$$||u||_{X^{s,b}} := ||S(-t)u(t)||_{H_t^b H^s} = \left|\left|\langle i(\partial_t - \mathcal{L})\rangle^b \langle i\nabla\rangle^s u\right|\right|_{L^2} \tag{0.2.13}$$

On voit que ce sont des espaces de type Sobolev adaptés au flot linéaire de (0.1.1). En particulier, l'estimation linéaire (0.2.1) dans ces espaces résulte d'une estimation générale sur les fonctions d'une variable du type

$$\left|\left|\psi_T \int_0^t g(t')\mathrm{d}t'\right|\right|_{H^b} \lesssim ||g||_{H^{b-1}} \tag{0.2.14}$$

où $\psi_T$ est une version lisse de l'indicatrice de $[0;T]$.

Dans le cas où $\mathcal{N}(u) = \partial_x f(u)$ et $f(u) = u^p$, on est donc ramenés à montrer une *estimation multilineaire* du type

$$\left|\left|\partial_x \prod_{i=1}^p u_i\right|\right|_{X^{s,b-1}} \lesssim \prod_{i=1}^p ||u_i||_{X^{s,b'}} \tag{0.2.15}$$

avec $b' \leqslant b$ pour conclure. On voit que la dérivée en $x$ est récupérée par le gain de régularité sur $b$. Une fois (0.2.15) prouvée, on peut conclure l'argument de point fixe, permettant ainsi de construire un flot *semi-linéaire*.

De nombreux outils pour prouver (0.2.15) ont été développés, par exemple en utilisant l'inégalité de Hölder puis en estimant chaque terme grâce aux estimations de Strichartz. Mais on peut aussi obtenir (0.2.15) directement, en exploitant les interactions entre plusieurs solutions linéaires, et on obtient alors de meilleurs estimations que le seul recours aux estimations de Strichartz (qui mesure seulement une solution linéaire). En particulier, Tao [Tao01] a fait une étude systématique de telles estimations et de leur optimalité dans plusieurs contextes. Lorsque $\mathcal{N}$ ne comporte pas de dérivée, la procédure précédente permet d'obtenir la propriété de contraction dans des espaces moins réguliers et également dans le cas de fonctions périodiques. Lorsque la non-linéarité comporte une dérivée, un des arguments clé pour la récupérer est un effet de lissage obtenu grâce à une relation algébrique du type

$$|\omega(\xi_1 + \xi_2) - \omega(\xi_1) - \omega(\xi_2)| \gtrsim |\xi_1 + \xi_2|^2 \tag{0.2.16}$$

(ici dans le cas d'une non-linéarité quadratique) où $\omega$ est le symbole de $\mathcal{L}$. Par exemple, une telle estimation permet d'obtenir le caractère semi-linéairement bien posé dans $H^s(\mathbb{R})$, $s > -3/4$ pour l'équation KdV [KPV96] et dans $L^2(\mathbb{T}^2)$ et $L^2(\mathbb{R}^2)$ pour l'équation KP-II [Bou93b].

Cependant, l'existence de *résonances* pour lesquelles la relation (0.2.16) n'est plus vraie n'est pas seulement une difficulté technique : Molinet, Saut et Tzvetkov [MST02b] ont montré qu'une interaction résonante basse fréquence-haute fréquence empêche le flot d'être régulier (au moins $\mathcal{C}^2$), ce qui implique que (0.2.1)-(0.2.2) ne peuvent avoir lieu pour *aucun* choix de $\mathbf{F}(T)$ et $\mathbf{N}(T)$. Il faut alors abandonner les méthodes de point fixe et revenir aux méthodes de type énergie et compacité comme pour le théorème général 0.2.2.

Comme expliqué ci-dessus, lorsque l'équation n'est pas semi-linéaire, on ne peut pas espérer montrer que le flot non-linéaire est une perturbation du flot linéaire sur un intervalle de temps fixé. Mais Koch et Tzvetkov [KT03] ont observé que pour une donnée localisée en fréquences, le flot agit linéairement sur des temps petits (de l'ordre d'une puissance négative de la fréquence). On obtient alors une version améliorée de (0.2.8) sur ces intervalles de temps. Bien sûr, le prix à payer est qu'il faut alors "empiler" ces estimations sur tous les petits intervalles de temps pour récupérer une estimation sur des temps $T = O(1)$. Mais en choisissant convenablement la taille des petits intervalles, on obtient des estimations a priori permettant d'utiliser un argument de compacité dans des espaces de régularité plus faible que dans la théorie générale de Kato. Cette nouvelle procédure consiste donc à trouver des espaces $\mathbf{F}(T)$, $\mathbf{N}(T)$ et $\mathbf{B}(T)$ dans lesquels on puisse obtenir les estimations

$$||u||_{\mathbf{F}(T)} \lesssim ||u||_{\mathbf{B}(T)} + ||\mathcal{N}(u)||_{\mathbf{N}(T)} \tag{0.2.17}$$

$$||\mathcal{N}(u)||_{\mathbf{N}(T)} \lesssim ||u||^p_{\mathbf{F}(T)} \tag{0.2.18}$$

$$||u||^2_{\mathbf{B}(T)} \lesssim ||u_0||^2_{H^s} + ||u||^{p+1}_{\mathbf{F}(T)} \tag{0.2.19}$$

On a vu plus haut que les espaces de Bourgain permettent de capter un gain de régularité par rapport à la méthode d'énergie standard. Il est alors naturel de chercher à bénéficier de l'efficacité des espaces de Bourgain sur ces petits intervalles de temps. Cette procédure a été formalisée par Ionescu, Kenig et Tataru [IKT08] dans le contexte de l'équation KP-I sur $\mathbb{R}^2$. L'idée est d'obtenir (0.2.17)-(0.2.18)-(0.2.19) avec des espaces $\mathbf{F}(T)$ et $\mathbf{N}(T)$ ayant une structure de type $X^{s,b}$ uniformément sur les intervalles de taille $N^{-\alpha}$ pour chaque composante dyadique $P_N u$ de la norme. On ne peut alors pas conclure directement par une méthode de point fixe puisqu'il faut d'abord empiler ces estimations comme dans la méthode précédente, ce qui fait apparaître l'espace d'énergie $\mathbf{B}(T)$. On utilise alors (0.2.17)-(0.2.18)-(0.2.19) pour obtenir une borne a priori sur les solutions et conclure par un argument de compacité. Notons qu'il faut de nouveau travailler (en utilisant l'argument de Bona-Smith) pour obtenir seulement la continuité du flot.

## 0.3 Présentation des travaux de thèse

### 0.3.1 Le problème de Cauchy pour l'équation KP-I

Dans cette section, on s'intéresse à l'équation KP-I

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{0.3.1}$$

Cette équation s'inscrit dans le formalisme général introduit au chapitre précédent. En effet, $\mathcal{L} = -\partial_x^3 + \partial_x^{-1}\partial_y^2$, où l'opérateur $\partial_x^{-1}$ est défini comme le multiplicateur de Fourier de symbole (singulier) $\dfrac{1}{i\xi}$ , et $\mathcal{N}(u) = \partial_x f(u) = \partial_x(-u^2/2)$.

**Résultats antérieurs**
Le problème de Cauchy pour cette équation a été étudié de manière intensive depuis plusieurs décennies. La méthode de Kato a été utilisée par plusieurs auteurs [Uka89, IMS92, Sau93, IMS95, IN98] pour obtenir le caractère bien posé local de (0.3.1) (et également pour une nonlinéarité avec une puissance plus générale) dans un sous-espace (adapté au symbole singulier de $\mathcal{L}$) de $H^s(\Omega)$, $s > 1 + d/2 = 2$ avec $\Omega = \mathbb{R}^2$ ou $\Omega = \mathbb{T}^2$. Comme on l'a vu à la section précédente, cette méthode n'est pas sensible à la forme particulière de l'opérateur $\partial_x^3 - \partial_x^{-1}\partial_y^2$. En particulier, elle s'applique également pour l'équation KP-II, pour laquelle l'opérateur linéaire est $\partial_x^3 + \partial_x^{-1}\partial_y^2$. Cette dernière équation a alors été étudiée par Bourgain [Bou93b], qui a montré que le problème de Cauchy est globalement semi-linéairement bien posé dans $L^2(\Omega)$ pour $\Omega = \mathbb{R}^2$ et $\Omega = \mathbb{T}^2$. La preuve de l'estimation bilinéaire (0.2.15) utilise de manière cruciale le signe de l'opérateur linéaire. Plus précisément, en notant

$$\omega_{II}(m,n) := m^3 - n^2/m$$

le symbole de l'opérateur linéaire pour l'équation KP-II, la relation algébrique

$$\omega_{II}(m_1+m_2,n_1+n_2) - \omega_{II}(m_1,n_1) - \omega_{II}(m_2,n_2) = \frac{m_1 m_2}{m_1+m_2}\left\{3(m_1+m_2)^2 + \left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right)^2\right\} \quad (0.3.2)$$

montre que les seules résonances sont triviales, et fournit ainsi un effet de lissage qui permet de récupérer la perte de dérivée dans la non-linéarité. Cette méthode a été rafinée dans plusieurs travaux [TT01, IM01, Had08] et culmine avec le résultat de Hadac, Herr et Koch [HHK09] qui montre le caractère bien posé global et la diffusion dans l'espace critique pour le changement d'échelle $\dot{H}^{-1/2,0}(\mathbb{R}^2)$. Le problème est également globalement bien posé dans $L^2(\mathbb{R}\times\mathbb{T})$ [MST11].

Pour l'équation KP-I (0.3.1), la situtation est différente : le symbole est maintenant

$$\omega_I(m,n) = m^3 + n^2/m$$

et le signe "+" entre les deux termes positifs de (0.3.2) devient un signe "-". Ainsi, il y a un ensemble assez large de fréquence réonantes, et l'estimation bilinéaire n'est alors plus vraie sur $\mathbb{R}^2$ [MST02b]. Molinet, Saut et Tzvetkov ont également montré que ces résonances empêchent le flot d'être régulier et ainsi, les méthodes de point fixe ne peuvent pas s'appliquer. De plus, Koch et Tzvetkov [KT08] ont montré que le problème n'est effectivement pas semi-linéaire au sens de la définition 0.1.3 en exhibant une famille de solutions qui empêche le flot d'être uniformément continu.

En utilisant alors la méthode d'énergie en temps petits présentée à la section précédente, Kenig [Ken04] puis Ionescu et Kenig [IK07] ont montré que le problème est globalement bien posé dans le second espace d'énergie

$$\mathbf{Z}^2(\Omega) := \left\{u_0 \in L^2,\ \partial_x u_0 \in L^2,\ \partial_x^{-1}\partial_y u_0 \in L^2,\ \partial_x^2 u_0 \in L^2,\ \partial_x^{-2}\partial_y^2 u_0 \in L^2\right\} \quad (0.3.3)$$

avec $\Omega = \mathbb{R}^2$ ou $\Omega \in \left\{\mathbb{R}\times\mathbb{T}, \mathbb{T}^2\right\}$ respectivement. En effet, l'espace (0.3.3) est lié à une quantité conservée par le flot : en plus du Hamiltonien

$$\mathcal{H}(u) = \frac{1}{2}\left|\left|\partial_x u\right|\right|_{L^2}^2 + \frac{1}{2}\left|\left|\partial_x^{-1}\partial_y u\right|\right|_{L^2}^2 - \frac{1}{6}\int_\Omega u^3 \mathrm{d}x\mathrm{d}y \quad (0.3.4)$$

la fonctionnelle

$$\mathcal{Z}(u) = \frac{3}{2}\left|\left|\partial_x^2 u\right|\right|_{L^2}^2 + \frac{5}{6}\left|\left|\partial_x^{-2}\partial_y^2 u\right|\right|_{L^2}^2 + 5\left|\left|\partial_y u\right|\right|_{L^2}^2 - \frac{5}{6}\int u^2(\partial_x^{-2}\partial_y^2 u) \\ - \frac{5}{6}\int u(\partial_x^{-1}\partial_y u)^2 + \frac{5}{4}\int u^2\partial_x^2 u + \frac{5}{24}\int u^4 \quad (0.3.5)$$

est également conservée par le flot sur $\mathbf{Z}^2$ ([MST02a]). Ainsi l'espace (0.3.3) permet de donner un sens à cette fonctionnelle et d'avoir encore équivalence entre localement et globalement bien posé. En fait, l'équation (0.3.1) admet une infinité de quantités formellement conservées, mais il n'est pas évident de trouver un cadre fonctionnel permettant de les justifier [MST02a]. La preuve de [Ken04, IK07] repose sur l'estimation de Strichartz localisée en fréquence et en temps petits

$$\left|\left|e^{t\mathcal{L}}P_M^x P_N^y u_0\right|\right|_{L^2(|t|\lesssim M^{-1}))L^\infty_{xy}} \lesssim \Lambda_\Omega(M,N)\left|\left|u_0\right|\right|_{L^2} \quad (0.3.6)$$

où $P_M^x$ et $P_N^y$ sont des projecteurs spectraux. Le coefficient $\Lambda_\Omega(M,N)$ dépend de la géométrie du domaine spatial :

$$\Lambda_\Omega(M,N) \lesssim \begin{cases} M^{0+} \text{ si } \Omega = \mathbb{R}^2 \\ (1\vee NM^{-2})^{1/2}M^{0+} \text{ si } \Omega = \mathbb{R}\times\mathbb{T} \\ (1\vee NM^{-2})^{1/2}M^{(3/8)+} \text{ si } \Omega = \mathbb{T}^2 \end{cases} \quad (0.3.7)$$

Notons que pour $\Omega = \mathbb{R}^2$, l'estimation (0.3.6) a lieu même sur des intervalles de temps de taille $O(1)$ et pour des fonctions non localisées en fréquence (avec une perte de dérivée pour le cas limite $L_t^2 L_{xy}^\infty$ [Sau93]).

(0.3.6) permet alors de contrôler $||\partial_x u||_{L^1_T L^\infty_{xy}}$, qui est la quantité clé dans l'estimation d'énergie classique (0.2.6).

Afin de diminuer le nombre de dérivées en $x$ requises par la méthode précédente, Ionescu, Kenig et Tataru ont alors introduit la dernière méthode de la section précédente et ont ainsi obtenu le caractère bien posé global dans l'espace d'énergie

$$\mathbf{E}(\mathbb{R}^2) := \left\{u_0 \in L^2, \partial_x u_0 \in L^2, \partial_x^{-1}\partial_y u_0 \in L^2\right\} \quad (0.3.8)$$

associé au Hamiltonien. Cette amélioration par rapport à la méthode précédente est l'analogue de la méthode de Bourgain par rapport à la méthode de point fixe dans les espaces de Strichartz : elle permet d'incorporer l'effet bilinéaire dans la non-linéarité, et d'obtenir ainsi une meilleure estimation qu'en utilisant seulement (0.3.6) et l'inégalité de Hölder.

Le point clé de la preuve de [IKT08] est une estimation bilinéaire pour la partie résonante de la non-linéarité :

$$\sup_{|I|\sim M^{-1}} ||\psi_I(t)\partial_x \mathfrak{R}(P_{M_1}u\cdot P_{M_2}v)||_{X^{0,-1/2}} \\ \lesssim (M_1\wedge M_2\wedge M)^{-1/2} \sup_{|I_1|\sim M_1^{-1}} ||\psi_{I_1}(t)P_{M_1}u||_{X^{0,1/2}} \sup_{|I_2|\sim M_2^{-2}} ||\psi_{I_2}(t)P_{M_2}v||_{X^{0,1/2}} \quad (0.3.9)$$

où le supremum est pris sur les intervalles de temps $I\subset[0;T]$, et $P_{M_i}$ est la projection sur les fréquences $|m|\sim M_i$ et $\mathfrak{R}$ est la projection de $P_{M_1}u\cdot P_{M_2}v$ sur l'ensemble des fréquences vérifiant

$$|\omega_I(m_1+m_2,n_1+n_2) - \omega_I(m_1,n_1) - \omega_I(m_2,n_2)| \lesssim |m_1 m_2(m_1+m_2)|$$

i.e où l'effet de lissage pour l'équation KP-II n'a pas lieu.

Lorsque $\Omega = \mathbb{T}^2$, Zhang [Zha15] a alors adapté la preuve de [IKT08] et a obtenu l'analogue de (0.3.9) mais avec une perte de dérivée :

$$\sup_{|I|\sim M^{-1}} ||\psi_I(t)\partial_x\Re(P_{M_1}u\cdot P_{M_2}v)||_{X^{0,-1/2}} \lesssim \sup_{|I_1|\sim M_1^{-1}} ||\psi_{I_1}(t)P_{M_1}u||_{X^{0,1/2}} \sup_{|I_2|\sim M_2^{-2}} ||\psi_{I_2}(t)P_{M_2}v||_{X^{0,1/2}} \quad (0.3.10)$$

ce qui restreint la régularité en $x$ et implique le caractère bien posé dans un espace strictement plus petit que l'espace d'énergie (à une différence logarithmique près).

**Le problème de Cauchy sur un cylindre**
Le principal résultat de cette thèse est le suivant :

**Théorème 0.3.1**

*Le problème de Cauchy pour (0.3.1) est globalement bien posé dans l'espace d'énergie* $\mathbf{E}(\mathbb{R}\times\mathbb{T})$.

La preuve de ce résultat fait l'objet du chapitre 3. Comme dans [IKT08], le point clé de la preuve est l'adaptation de (0.3.9) dans le cas $\Omega = \mathbb{R}\times\mathbb{T}$, sans perte de dérivée. Pour cela, il faut adapter la preuve de (0.3.9) de manière analogue à [Zha15], mais en tirant parti du fait que la variable de Fourier correspondant à la direction principale de propagation vit dans $\mathbb{R}$ (et non dans $\mathbb{Z}$), ce qui permet de mesurer précisément les petites variations des lignes de niveaux de la fonction de résonance. Notons que dans le cas où cette fréquence est un entier, la mesure de comptage est trop grossière pour capter ces petites variations. Dans le cas non résonant, il faut également adapter l'estimation de Strichartz usuelle [Sau93] : on obtient l'analogue pour des fonctions localisées en fréquences et en temps petits mais avec une légère perte de dérivée qui nécessite une petite modification des espaces de Ionescu, Kenig et Tataru. Enfin, remarquons qu'un argument supplémentaire permet d'obtenir l'unicité dans l'espace dans lequel on construit la solution, et non uniquement au sens de limite de solutions régulières.

**A propos du soliton de KdV**
L'équation (0.3.1) est connue pour admettre des solitons 2D (des ondes progressives localisées) [dBS97], mais également les solitons 1D de l'équation KdV dont (0.3.1) est une extension à deux dimensions. Contrairement aux "lumps", les solitons 1D

$$u_c(t,x,y) = Q_c(x-ct) \text{ avec } Q_c(x) := 3c\cdot\cosh\left(\frac{\sqrt{c}}{2}x\right)^{-2} \quad (0.3.11)$$

ne sont pas localisés dans la direction transverse, et n'appartiennent donc pas à l'espace d'énergie $\mathbf{E}(\mathbb{R}^2)$, ce qui motive l'étude de (0.3.1) sur $\mathbb{R}\times\mathbb{T}$. La question de la stabilité orbitale de ces solutions particulières a déjà été étudiée par Rousset et Tzvetkov [RT12] : en utilisant l'étude spectrale de [APS97] et en reprenant le schéma de preuve de [Ben72] (voir [GSS87] pour un argument général), ils ont ainsi obtenu la stabilité orbitale de $(u_c)$ pour les petites vitesses $0 < c < 4/\sqrt{3}$ et pour des perturbations dans le second espace d'énergie $\mathbf{Z}^2(\mathbb{R}\times\mathbb{T})$ (0.3.3). L'argument de [RT12] n'utilisant que l'Hamiltonien de (0.3.1), il est directement transposable à des solutions $u\in\mathbf{E}(\mathbb{R}\times\mathbb{T})$. Ainsi, le flot construit au chapitre suivant permet d'obtenir le

**Corollaire 0.3.2**

*Le soliton de KdV est orbitalement stable dans l'espace d'énergie $\mathbf{E}(\mathbb{R}\times\mathbb{T})$ pour les petites vitesses $0 < c < 4/\sqrt{3}$.*

**Sur la régularité du flot**

Molinet, Saut et Tzvetkov [MST02b] ont montré que le flot pour (0.3.1) sur $\mathbb{R}^2$ ne pouvait pas être de classe $\mathcal{C}^2$. En exploitant le même phénomène de résonance, on a un résultat analogue sur $\mathbb{R}\times\mathbb{T}$ :

**Proposition 0.3.3**

*Soient $(s_1, s_2) \in \mathbb{R}^2$. Alors il n'existe pas de temps $T > 0$ tel que le flot engendré par l'équation (0.1.4) $\Phi_t : \varphi \mapsto u(t)$, $t \in [-T; T]$, soit $\mathcal{C}^2$-différentiable en zéro de*

$$H^{s_1,s_2}(\mathbb{R}\times\mathbb{T}) := \left\{u_0 \in L^2(\mathbb{R}\times\mathbb{T}),\ |D|_x^{s_1} u_0 \in L^2(\mathbb{R}\times\mathbb{T}),\ |D|_y^{s_2} u_0 \in L^2(\mathbb{R}\times\mathbb{T})\right\}$$

*dans $H^{s_1,s_2}(\mathbb{R}\times\mathbb{T})$. On a un résultat analogue sur $\mathbf{E}(\mathbb{R}\times\mathbb{T})$.*

Ainsi, cette proposition motive l'abandon des méthodes de point fixe (qui fournissent un flot analytique) et le recours aux méthodes d'énergie raffinées. En utilisant l'argument de Bona et Smith [BS75], le flot construit au chapitre suivant est alors continu. En adaptant un résultat de Koch et Tzvetkov [KT08], on montre alors que le problème de Cauchy pour (0.3.1) sur $\mathbb{R}\times\mathbb{T}$ n'est effectivement pas semi-linéaire au sens de la définition 0.1.3 :

**Théorème 0.3.4**

*Le flot pour l'équation (0.3.1) n'est pas uniformément continu sur les bornés de $\mathcal{C}([-1; 1], \mathbf{E}(\mathbb{R}\times\mathbb{T}))$.*

## 0.3.2 L'équation KP-I d'ordre 5

On s'intéresse maintenant à l'équation KP-I d'ordre 5

$$\partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{0.3.12}$$

**Un comportement différent selon le domaine spatial**

Le problème de Cauchy pour (0.3.12) a été précédemment étudié par Saut et Tzvetkov [ST00, ST01]. En utilisant la méthode de Bourgain, ils ont ainsi obtenu que (0.3.12) est globalement semi-linéairement bien posée dans les espaces d'énergie $\mathbf{E}^2(\mathbb{R}^2)$ et $\mathbf{E}^2(\mathbb{T}\times\mathbb{R})$ associés au Hamiltonien de (0.3.12). Cependant, en utilisant une interaction basse fréquence - haute fréquence résonante, les auteurs ont montré que l'estimation bilinéaire ne peut avoir lieu dans aucun espace de Bourgain lorsque (0.3.12) est posée sur $\mathbb{T}^2$. Comme pour l'équation KP-I standard, cette obstruction n'est pas seulement technique, puisqu'on a la

**Proposition 0.3.5**

*Soient $(s_1, s_2) \in \mathbb{R}^2$. Alors il existe une période $\lambda > 0$ telle pour tout $T > 0$, le flot $\Phi_{t,\lambda}$ engendré par (0.3.12) n'est pas $\mathcal{C}^2$-différentiable en zéro de $H^{s_1,s_2}(\mathbb{T}^2_\lambda)$ dans $H^{s_1,s_2}(\mathbb{T}^2_\lambda)$, où $\mathbb{T}^2_\lambda = \mathbb{T}\times\lambda^{-1}\mathbb{T}$. De plus, on a un résultat analogue sur $\mathbf{E}^2(\mathbb{T}^2_\lambda)$.*

Ainsi, au moins pour certains choix de période, les méthodes de point fixe ne peuvent pas s'appliquer. Contrairement à l'équation KP-I standard, qui est quasi-linéaire quel que soit le choix du domaine spatial, l'équation (0.3.12) présente un comportement qualitatif radicalement différent selon que la donnée est périodique dans la direction transverse ou non.

**Résultats sur le problème quasi-linéaire**
En utilisant la méthode d'énergie en temps petits, Ionescu et Kenig [IK07] ont montré que le problème de Cauchy pour (0.3.12) est globalement bien posé dans l'espace d'énergie $\mathbf{E}^2(\mathbb{R}\times\mathbb{T})$. Ce résultat repose sur l'estimation de Strichartz en temps petits

$$||e^{t\mathcal{L}}P_M^xP_N^yu_0||_{L^2(|t|\lesssim M^{-2})L^\infty_{xy}} \lesssim (1\vee NM^{-3})^{1/2}M^{(-1/2)+}\,||u_0||_{L^2} \tag{0.3.13}$$

Dans le cas ou la donnée est également périodique en $x$, il faut remplacer l'estimation d'intégrale oscillante de [IK07] par une estimation sur une somme exponentielle : on obtient alors l'analogue de (0.3.13) mais avec une perte de dérivée

$$||e^{t\mathcal{L}}P_M^xP_N^yu_0||_{L^2(|t|\lesssim M^{-2})L^\infty_{xy}} \lesssim (1\vee NM^{-3})^{1/2}M^{(15/32)+}\,||u_0||_{L^2} \tag{0.3.14}$$

En utilisant cette estimation dans la méthode d'énergie en temps petits, on obtient alors la condition $s>2+15/32$ qui est plus restrictive que la méthode d'énergie standard. Cette obstruction vient du fait que l'ordre élevé de l'opérateur $\mathcal{L}$ apporte un meilleur effet de dispersion seulement lorsque la variable $x$ vit sur la droite $\mathbb{R}$ (ce qui est quantifié grâce au lemme de Van Der Corput) et que cet effet n'est plus présent si la donnée est périodique en $x$ (car les sommes exponentielles décroissent plus lentement).

Pour construire un flot sur un espace plus large que $H^s(\mathbb{T}^2)$, $s>2$, il faut alors utiliser la méthode de Bourgain en temps petits. La méthode s'appuie sur l'estimation de la partie résonante

$$\begin{aligned}\sup_{|I|\sim M^{-2}}&\,||\psi_I(t)\partial_x\mathfrak{R}(P_{M_1}u\cdot P_{M_2}v)||_{X^{0,-1/2}}\\ &\lesssim \Gamma_{\mathbb{T}^2}(M_1,M_2,M)\sup_{|I_1|\sim M_1^{-2}}||\psi_{I_1}(t)P_{M_1}u||_{X^{0,1/2}}\sup_{|I_2|\sim M_2^{-2}}||\psi_{I_2}(t)P_{M_2}v||_{X^{0,1/2}}\end{aligned} \tag{0.3.15}$$

où dans ce cas on travaille sur des intervalles de temps de taille $M^{-2}$, et

$$\Gamma_{\mathbb{T}^2}=(M_1\wedge M_2\wedge M)^{-1/2}$$

Dans le cas de $\mathbb{R}^2$, Guo, Huo et Fang [GHF17] ont montré que cette estimation est valable avec

$$\Gamma_{\mathbb{R}^2}(M_1,M_2,M)=(M_1\wedge M_2\wedge M)^{-1/2}(M_1\vee M_2\vee M)^{-3/2}$$

On voit que sur le tore on a une perte de dérivée similaire à celle dans le cas de l'équation KP-I standard. Néanmoins, (0.3.15) est suffisante pour obtenir le

**Théorème 0.3.6**

*L'équation (0.3.12) est globalement bien posée dans l'espace d'énergie* $\mathbf{E}^2(\mathbb{T}^2)$.

Bien qu'énoncé et démontré au chapitre 4 sur un tore carré, la preuve de ce théorème s'adapte de manière directe au cas d'un tore de périodes quelconques, il n'y a donc pas de contradiction avec la proposition précédente. On peut alors se demander si le problème est effectivement quasi-linéaire au sens de la définition 0.1.3. Notons qu'un argument clé dans les préliminaires de la preuve consiste à travailler avec des solutions à moyenne nulle en $x$. On peut toujours se ramener à ce cas puisque d'une part la moyenne en $x$ d'une solution est constante (en $t$ et $y$) et d'autre part la transformation

$$T_\theta := u(t,x,y)\mapsto u(t,x+\theta t,y)-\theta$$

laisse l'équation invariante, ce qui réduit le problème en prenant $\theta = \int_{\mathbb{T}} u_0(x,y)\mathrm{d}x$. Or la transformation $T_\theta$ n'est *pas* uniformément continue. La preuve de [KT08] et du théorème 0.3.4 repose d'ailleurs sur l'utilisation d'une version localisée (dans le cas de variables non périodiques) de cette transformation. Cet argument est très général : ainsi, même pour l'équation KdV, le flot n'est pas uniformément continu car également préservé par cette transformation [KT06]. Or, sur $L^2(\mathbb{T})$, cette équation est connue pour être semi-linéairement bien posée sur l'hyperplan des données à moyenne nulle, par le travail de Bourgain [Bou93a]. Ainsi, une vraie manifestation du caractère non semi-linéaire du problème serait de montrer que le flot n'est pas uniformément continu même sur les hyperplans de données à moyenne fixée.

# Chapter 1

# Overview of some methods for solving the Cauchy problem for dispersive PDEs

This chapter contains an extended version of chapter 0 sections 0.1-0.2.

## 1.1 Preliminaries

The aim of this thesis is to develop on the low regularity well-posedness theory for some nonlinear dispersive PDEs exhibiting a quasilinear behaviour.

We will consider evolution equations having the form

$$\partial_t u = \mathcal{L}u + \mathcal{N}(u) \tag{1.1.1}$$

We restrict ourself to scalar equations $u : (t, \mathbf{x}) \in \mathbb{R} \times \Omega \mapsto u(t, \mathbf{x}) \in \mathbb{R}$. Here $u(t, \cdot)$ lies in a functional space $X$, $\Omega$ is a domain in $\mathbb{R}^d$ (complemented with periodic boundary conditions in the case of a domain with boundary), $\mathcal{L}$ is a skew-symmetric linear operator on $X$, and $\mathcal{N}$ is the nonlinearity. Such equations arise frequently in the study of wave-like phenomena, in particular as asymptotic models in fluid mechanics. In this thesis, we consider KP type equations in which $d = 2$, $\Omega = \mathbb{R} \times \mathbb{T}$ or $\Omega = \mathbb{T}^2$ (where $\mathbb{T}^d$ is the $d$ dimensional torus) and the linear and nonlinear operators take the form

$$\mathcal{L} = \partial_x A \tag{1.1.2}$$

and

$$\mathcal{N}(u) = \partial_x f(u) \tag{1.1.3}$$

with $A$ being a symmetric (pseudo) differential operator and $f : u \in \mathbb{R} \mapsto u^p$ where $p \in \mathbb{N}$ with $p > 1$.

### 1.1.1 Modeling

In this subsection, we briefly indicate how (1.1.1) appears in fluid mechanics. We follow the book [Lan13].

The Kadomtsev-Petviashvili equations have been originally derived in 1970 [KP70] in order to describe the time evolution of the surface of an incompressible, irrotational, inviscid shallow fluid on a flat bottom with small amplitude, long wavelength oscillations and weak dependence in the transverse direction. We will take into account both gravity $g$ and surface tension $\sigma$.

Defining $h_0$ (respectively $a$, $L_x$, $L_y$) to be the characteristic depth (respectively amplitude, longitudinal scale, transverse scale), we then introduce the relevant shallowness, nonlinear and transversality parameters $\delta := h_0/L_x$, $\varepsilon := a/h_0$ and $\gamma = L_x/L_y$. Then the aforementioned regime corresponds to $\delta, \varepsilon, \gamma \to 0$. In order to get a model having both dispersion and nonlinearity, we will then consider the particular regime

$$\varepsilon \ll 1,\ \delta = \gamma = \sqrt{\varepsilon} \tag{1.1.4}$$

Starting from Euler equations, after nondimensionalisation and scaling we get that the free surface $z = \eta(t, x, y)$ satisfies, in the limit $\varepsilon \to 0$, the wave equation

$$\partial_t^2 \eta - \partial_x^2 \eta = 0$$

Thus it decouples into two components $\eta_\pm$ going respectively to the left and to the right, and at order 1 in $\varepsilon$ the wave going to the right satisfy

$$\partial_t \eta_+ + \partial_x \eta_+ + \frac{\varepsilon}{2}\left(\partial_x^{-1}\partial_y^2 \eta_+ + \left(\frac{1}{3} - b\right)\partial_x^3 \eta_0 + 3\eta_+ \partial_x \eta_+\right) = o(\varepsilon) \tag{1.1.5}$$

where $b := \sigma/(\rho g h_0^2) = \dfrac{B}{\varepsilon}$ and $B = \sigma/(\rho g L_x^2)$ is the Bond number (and $\rho$ is the density, assumed to be constant).

It is worth noticing that the surface tension has a strong effect on the dispersion. We can then distinguish three cases depending on the physical properties of the fluid :

$$b < \frac{1}{3}, b > \frac{1}{3} \text{ and } b = \frac{1}{3}$$

In the first two regimes, since the constant in front of the dispersive term is non zero, we can perform a last change of variables and rescaling to get the canonical forms

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \ (b > 1/3 : \text{ KP-I equation}) \tag{1.1.6}$$

$$\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \ (b < 1/3 : \text{ KP-II equation}) \tag{1.1.7}$$

The last regime corresponds to $b \approx 1/3$. For this critical value of the rescaled Bond number, in order to keep dispersion in the model, we need to work in a less nonlinear regime, replacing (1.1.4) with

$$\varepsilon \ll 1,\ \delta = \varepsilon^{1/4},\ \gamma = \sqrt{\varepsilon} \tag{1.1.8}$$

and $b \sim 1/3$ with $b = 1/3 + \beta\varepsilon$. The component going to the right satisfies at order 1 in $\varepsilon$

$$\partial_t \eta_+ + \partial_x \eta_+ + \frac{\varepsilon}{2}\left(\partial_x^{-1}\partial_y^2 \eta_+ - \beta\partial_x^3 \eta_+ + \frac{1}{45}\partial_x^5 \eta_+ + 3\eta_+\partial_x \eta_+\right) = o(\varepsilon) \tag{1.1.9}$$

Thus taking $\beta = 0$ and with another change of variables and rescaling, we obtain the canonical form of the fifth-order KP-I equation

$$\partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{1.1.10}$$

### 1.1.2 Hamiltonian structure of the equation

As we required the operators $\mathcal{L}$ and $\mathcal{N}$ to have the particular form (1.1.2)-(1.1.3) then (1.1.1) possesses a Hamiltonian structure :

$$\partial_t u = J \cdot \nabla \mathcal{H}(u) \tag{1.1.11}$$

Here the symplectic operator is given by $J = \partial_x$, and the Hamiltonian functional is defined as

$$\mathcal{H}(u) = \frac{1}{2}\langle Au, u\rangle + \frac{1}{(p+1)}\int_\Omega u(x)^{p+1}\mathrm{d}x, \tag{1.1.12}$$

where $\langle\cdot\rangle$ is the scalar product in $L^2(\Omega)$.

Now, for an equation under the form (1.1.11), the Hamiltonian (also called energy) $\mathcal{H}$ is (at least formally) conserved by the flow.

There is actually another conservation law for the equation : if we multiply it by $u$ and then integrate over $\Omega$, using that $\mathcal{L}$ is skew-adjoint and that $f$ is smooth and real-valued, we get that the $L^2(\Omega)$ norm of $u$ is also an invariant of the equation.

This means that in both cases $X = L^2(\Omega)$ or $X = D(A)$ (when $A$ is semibounded from below, and $D(A)$ is its form domain), the corresponding conservation law makes sense, and allows one to have an a priori control on the corresponding norm (with the help of some Sobolev type inequalities to control the superquadratic terms $F(u)$ in $\mathcal{H}$), which allows one to extend the solution to any time $T > 0$.

### 1.1.3 Well-posedness, semilinear and quasilinear equations

Once we have a model of type (1.1.1), we are looking to solve the Cauchy problem

$$\begin{cases} \partial_t u = \mathcal{L}u + \mathcal{N}(u) \\ u(t=0) = u_0 \in X \end{cases} \tag{1.1.13}$$

where $u_0$ is given, belonging to a function space $X$.

**Definition 1.1.1**

*The Cauchy problem (1.1.13) is said* locally well-posed *in $X$ if the following conditions are satisfied :*

*(a) for any bounded set $B \subset X$, there exists a time $T > 0$ and a Banach space $\mathbf{X}_T$ continuously embedded in $\mathcal{C}([0;T], X)$ such that for every $u_0 \in B$ there exists a unique solution $u \in \mathcal{C}([0;T], X)$ satisfying (1.1.1) on $[0;T]$*

*(b) the flow map $\Phi : u_0 \in X \mapsto u \in \mathbf{X}_T$ is continuous*

**Remarks 1.1.2.**

*(i) If $\mathbf{X}_T = \mathcal{C}([0;T], X)$, (1.1.13) is said* unconditionnally *well-posed.*

*(ii) The notion that we are more interested in is that if (a)-(b) stands for any choice of $T > 0$, (1.1.13) is said* globally *well-posed.*

*(iii) It is important to notice that in the definition above, we require $T$ to depend only on $B$ (that is, on $||u_0||_{H^s}$) and not on the particular profile of the data. This is because we will deal with subcritical problems : for critical ones, it is more involved.*

*(iv) Sometimes, another condition is required : the persistence of regularity. This means that if $X = X^s$ belongs to some Banach scale $(X^s)$, then for any $s' \geqslant s$, if $u_0 \in X^{s'} \subset X^s$ then $u(t)$ stays in $X^{s'}$. This sometimes leads to a slightly weaker definition of uniqueness, in the sense of the unique limit of smooth solutions. For the models we consider, we will have both properties.*

The purpose of this thesis is then to show that the Cauchy problem for the models of subsection 1.1.1 are globally well-posed in some suitable function space $X$. As observed in the previous subsection, if we are able to get local well-posedness in spaces $X$ such that the conservation laws make sense, but large enough such that these conservation laws control the norm in $X$, then we get global well-posedness in $X$.

So, how large can we expect $X$ to be to still get local well-posedness ? Note that, besides the conservation laws of the previous subsection, when $\Omega = \mathbb{R}^d$ equations of type (1.1.1) generally possess a scale invariance : if $u$ is a solution, so is

$$u_\lambda(x,t) := \lambda^\gamma u(\lambda^\alpha x, \lambda^\beta t) \tag{1.1.14}$$

for some appropriate choice of exponents $\alpha, \beta, \gamma$ (depending on $\mathcal{L}$ and $f$). In particular, computing the $\dot{H}^s$ norm of $u_\lambda$, one can define the *scaling regularity* $s_c := \gamma/\alpha - 1/2$ for which $||u_\lambda||_{\dot{H}^{s_c}} = ||u||_{\dot{H}^{s_c}}$ for any $\lambda > 0$. Thus, for regularity below $s_c$, rescaling induces a growth of norms as $\lambda \to +\infty$, so it is expected that the Cauchy problem is ill-posed.

In order to prove local well-posedness above the scaling regularity, one can mimic the Cauchy theory in finite dimension and look at the integral formulation of (1.1.1)

$$u(t) = \mathrm{e}^{t\mathcal{L}} u_0 + \int_0^t \mathrm{e}^{(t-t')\mathcal{L}} \mathcal{N}(u(t'))\mathrm{d}t' \tag{1.1.15}$$

The Duhamel formula above means that in order to solve (1.1.1) one can look for a fixed point of the right-hand side of (1.1.15). If one succeeds to do so, then one can reasonably hope to recover a flow map which is at least Lipschitz, since the nonlinear function $f$ is assumed to be smooth. This motivates the following stronger notion of well-posedness (see [Tzv04]) :

**Definition 1.1.3**

*The Cauchy problem (1.1.13) is said to be* semilinearly well-posed *if it is well-posed in the sense of definition 1.1.1 and moreover the flow map $\Phi : u_0 \in X \mapsto u \in \mathbf{X}_T$ is* uniformly continuous *on bounded sets of $X$.*

**Remark 1.1.4.** *When the flaw map is not uniformly continuous, the problem is said to be* quasilinear.

However, there has been several examples of semilinear ill-posedness. For energy-supercritical models, this has been investigated by Lebeau [Leb05] in the context of the nonlinear wave equations (see also [AC09] for the nonlinear Schrödinger equation). Concerning one dimensional models with general power nonlinearity $f(u) = u^p$ (nonlinear Scrödinger equation, generalized KdV equation), it has been proved that there is semilinear ill-posedness below the scaling regularity (for large $p$) in the focusing cases[BKP$^+$96] by using solitary waves, and then even *above* [KPV01] and in the defocusing cases [CCT03]. Still, the Cauchy problem for these equations is semilinearly well-posed at higher regularity, so the quasilinear behaviour only manifests at very low regularity. In contrast, the problem for the KP-I equation on $\mathbb{R}^2$ has been shown by Molinet, Saut et Tzvetkov [MST02b] to be quasilinear at any regularity, which requires a different approach to the problem.

The situation is again different in the case of fully periodic equations. For example, in the case of the KdV equation on the circle, the flow map is not uniformly continuous on any ball of $H^{-1}(\mathbb{T})$ [KT06], yet it is on the hyperplanes of data with prescribed mean value. This is because the Galilean transform

$$u(t,x) \mapsto u\left(t, x + t\int_{\mathbb{T}} u\mathrm{d}x\right) - \int_{\mathbb{T}} u\mathrm{d}x \tag{1.1.16}$$

itself is not uniformly continuous.

In the next section, we will see different methods to prove that (1.1.13) is locally well-posed, and in particular how to deal with quasilinear equations.

## 1.2 Solving the Cauchy problem for some nonlinear dispersive PDEs

The study of (1.1.13) has gathered the attention of researchers for a few decades now, and different methods to get local well-posedness at lower and lower regularity have been developped.

### 1.2.1 Semilinear well-posedness at high regularity without derivative in the nonlinearity

In order to construct a solution by using an iteration scheme based on the Duhamel formula (1.1.15), the general strategy is to get two abstract estimates : a linear one

$$||u||_{\mathbf{F}(T)} \lesssim ||u_0||_{H^s} + ||(\partial_t - \mathcal{L})u||_{\mathbf{N}(T)} \tag{1.2.1}$$

and a nonlinear one

$$||\mathcal{N}(u) - \mathcal{N}(v)||_{\mathbf{N}(T)} \lesssim ||u - v||_{\mathbf{F}(T)} \left(||u||_{\mathbf{F}(T)} + ||v||_{\mathbf{F}(T)}\right)^{p-1}. \tag{1.2.2}$$

for some functional spaces $\mathbf{F}(T) \subset \mathcal{C}([0;T], H^s)$ and $\mathbf{N}(T)$ (and the contraction property is then obtained either with a gain of a factor $T^{0+}$ in (1.2.1) or (1.2.2), or with a smallness assumption on the data).

The simpler setting to try to apply this strategy is when there is no derivative in the nonlinearity and for a high enough regularity. Let us define the *Duhamel operator*

$$\mathfrak{D} : u \mapsto \int_0^t \mathrm{e}^{(t-t')\mathcal{L}} u(t')\mathrm{d}t' \tag{1.2.3}$$

Then, with the assumptions on $f$, $\mathcal{N}$ is continuous from $H^s$ to $H^s$ when $s > d/2$ and we have the estimate

$$\begin{aligned} ||f(u) - f(v)||_{H^s} \lesssim C\left(p, ||u||_{L^\infty}, ||v||_{L^\infty}\right) \{ &||u - v||_{H^s} \left(||u||_{L^\infty} + ||v||_{L^\infty}\right) \\ &+ ||u - v||_{L^\infty} \left(||u||_{H^s} + ||v||_{H^s}\right)\}, \end{aligned} \tag{1.2.4}$$

which holds for $s \geqslant 0$ and $u, v \in H^s \cap L^\infty$, so that the condition $s > d/2$ comes from the Sobolev embedding $H^s \hookrightarrow L^\infty$.

Using that $\mathrm{e}^{t\mathcal{L}}$ is unitary in $H^s$ on top of that, we readily obtain that $\mathfrak{D}$ is also bounded from $L^\infty(\mathbb{R}, H^s)$ to $L^\infty(\mathbb{R}, H^s)$, that is (1.2.1)-(1.2.2) hold with $\mathbf{F}(T) = \mathbf{N}(T) = \mathcal{C}([0;T], H^s)$ which implies that

$$\mathcal{X}_T := u \mapsto \mathrm{e}^{t\mathcal{L}} u_0 + \mathfrak{D} \circ \mathcal{N}(u) \tag{1.2.5}$$

is a contraction mapping on a closed ball of $\mathcal{C}([0;T],H^s)$ for a $T>0$ depending only on $||u_0||_{H^s}$, so existence follows. Uniqueness in the whole class $\mathcal{C}([0;T],H^s))$ is a consequence of the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\,||u-v||_{L^2}^2 \lesssim \langle u-v, f(u)-f(v)\rangle_{L^2} \lesssim C(||u||_{L^\infty}\,, ||v||_{L^\infty})\,||u-v||_{L^2}^2 \tag{1.2.6}$$

where the first inequality uses the skew-symmetry of $\mathcal{L}$ and the second one follows from (1.2.4). All in all, we arrive at the

**Theorem 1.2.1**

*The problem (1.1.13) with $\mathcal{N}=f$ is locally semilinearly well-posed on $H^s$ for any $s>d/2$.*

The argument above relies on the continuity of $\mathcal{N}$ from $H^s$ to $H^s$ for $s>d/2$, thus we have to modify it when there is a derivative in the nonlinearity.

### 1.2.2 Kato's theory for quasilinear equations

Another approach is to look at an approximate equation

$$\begin{cases}\partial_t u^\varepsilon = \mathcal{L}^\varepsilon u^\varepsilon + \mathcal{N}^\varepsilon(u^\varepsilon)\\ u^\varepsilon(t=0)=u_0^\varepsilon\end{cases} \tag{1.2.7}$$

for which it is easy to get a unique solution $u^\varepsilon$ and then to recover a unique solution to (1.1.13) by letting $\varepsilon\to 0$. In view of the previous remark, the idea is then to choose the regularization procedure such that $\mathfrak{D}\circ\mathcal{N}^\varepsilon$ is continuous from $L^\infty(\mathbb{R},H^s)$ to $L^\infty(\mathbb{R},H^s)$. This can be performed in several ways : a common one is to take $\mathcal{L}^\varepsilon=\mathcal{L}+\varepsilon\Delta$ in order to take advantage of the smoothing effect of the heat kernel. Indeed, the Duhamel formula for (1.2.7) reads

$$u^\varepsilon(t)=\mathrm{e}^{t\mathcal{L}^\varepsilon}u_0+\int_0^t \mathrm{e}^{(t-t')\mathcal{L}^\varepsilon}\mathcal{N}(u^\varepsilon(t'))\mathrm{d}t'$$

and now $\mathrm{e}^{t\mathcal{L}^\varepsilon}$ is no more unitary but a contraction semigroup instead, and is infinitely smoothing. This latter property allows us to check that the same argument as above applies, thus we get existence and uniqueness of $u^\varepsilon\in\mathcal{C}([0;T_\varepsilon],H^s)$ solution of (1.2.7). In order to pass to the limit $\varepsilon\to 0$, we derive the energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\,||u^\varepsilon(t)||_{H^s}^2 \lesssim \langle u^\varepsilon(t),\mathcal{N}^\varepsilon(u^\varepsilon(t))\rangle_{H^s} \lesssim ||\nabla u^\varepsilon(t)||_{L^\infty}\,C(||u^\varepsilon(t)||_{L^\infty})\,||u^\varepsilon(t)||_{H^s}^2 \tag{1.2.8}$$

where we have used that $\mathcal{L}^\varepsilon$ is maximally dissipative, and the last inequality follows by integrations by parts, (1.2.4) and the commutator estimate

**Lemma 1.2.2 (*[KP88]*)**

*Let $s\geqslant 0$ and $f,g\in H^s\cap L^\infty$, then*

$$||[\langle D\rangle s, f]g||_{L^2} \lesssim ||f||_{H^s}\,||g||_{L^\infty} + ||\nabla f||_{L^\infty}\,||g||_{H^{s-1}} \tag{1.2.9}$$

Thus the use of Gronwall's lemma and the Sobolev embedding $H^s\hookrightarrow W^{1,\infty}$ for $s>1+d/2$ provide that for a small $T$ uniform in $\varepsilon$, the family of solutions $(u^\varepsilon)$ is bounded in $\mathcal{C}([0;T],H^s)$. Moreover, using again (1.2.7) we get that $u^\varepsilon$ is also bounded in $W^{1,1}([0;T],H^{-m})$ for $m$ large enough. With both bounds we recover a strong limit in $\mathcal{C}([0;T],H^{-m}_{loc})$ which coincides with the weak limit in $L^\infty([0;T], H^s)$, and this limit is a solution of (1.1.1) in distributional sense. The continuity in time at the $H^s$ level follows from the energy estimate (1.2.8) for the limit, and the uniqueness from the energy estimate for the difference equation.

The continuity of the flow map follows from an argument due to Bona and Smith [BS75] : if we have data $u_{0,n} \to u_0$ in $H^s$ with corresponding solutions $u_n$ and $u$, then introduce a frequency approximation of identity $P_\varepsilon$ and denote $u_n^\varepsilon$ and $u^\varepsilon$ the solutions issued from $P_\varepsilon u_{0,n}$ and $P_\varepsilon u_0$. Then from the same energy estimate as above, one can take a common time of existence to all these solutions, and using the energy estimate for the difference equation, one can control terms $||v^\varepsilon - v||_{H^s} \lesssim C(R)\,||v_0^\varepsilon - v_0||_{H^s}$ where $R$ is the radius of a ball in $H^s$ containing all the solutions on the common time of existence, and $v$ and $v^\varepsilon$ are any of the solutions above. Thus, using triangle inequality to insert terms with $\varepsilon$, one can take $\varepsilon$ small enough and $n$ large enough to make $||u_n - u||_{L_T^\infty H^s}$ small, hence the continuity of the flow.

Thus we recover the general

**Theorem 1.2.3**

*The Cauchy problem (1.1.13) is locally well-posed in $H^s$ for $s > 1 + d/2$.*

Note that, although we had semilinear well-posedness in absence of derivative in $\mathcal{N}$, the limiting procedure does not allow us to recover it when $\mathcal{N} = \partial_x f$, and we must even use the extra argument described above to get only continuity of the flow map.

A crucial remark though is that both theorems 1.2.1 and 1.2.3 work equally well for a maximally dissipative operator $\mathcal{L}^\varepsilon$ and do not take into account all the power of the skew-symmetry of $\mathcal{L}$. This will be investigated in the next subsection.

### 1.2.3 A fixed point argument in Strichartz spaces

Let us first consider the case of a nonlinearity without derivative. To provide a concrete example, we shall study the Cauchy problem for the cubic nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u = |u|^2 u,\ \ (t,x) \in \mathbb{R}\times\mathbb{R} \tag{1.2.10}$$

It does not fit exactly into the abstract setting defined at the begining of this section since we choose to consider only real-valued equations, yet it will allow us to illustrate the method.

As we explained above, the idea to get below the regularity threshold $s > 1/2$ of theorem 1.2.1 is to measure more accurateley the oscillations in the linear evolution $\mathrm{e}^{it\partial_x^2}$. In our example, the kernel of the unitary group can be computed exactly, which shows that the free solutions enjoy the time decay property

$$\left|\left|\mathrm{e}^{it\partial_x^2} u_0\right|\right|_{L^\infty} \lesssim |t|^{-1/2}\,||u_0||_{L^1} \tag{1.2.11}$$

Thus, with a duality ($TT^\star$) argument and the use of Hardy-Littlewood-Sobolev inequality, we obtain the so-called *Strichartz estimate*

$$\left|\left|\mathrm{e}^{it\partial_x^2} u_0\right|\right|_{L_t^4 L^\infty} \lesssim ||u_0||_{L^2} \tag{1.2.12}$$

and its inhomogeneous version

$$\left|\left|\int_0^t \mathrm{e}^{i(t-t')\partial_x^2} u(t')\mathrm{d}t'\right|\right|_{L_t^4 L^\infty} \lesssim ||u||_{L_t^{4/3} L^1} \tag{1.2.13}$$

This means that the linear flow is not only unitary from $H^s$ to $L_t^\infty H^s$ but also bounded from $L^2$ to $L_t^4 L^\infty$. Therefore, (1.2.12)-(1.2.13) imply that (1.2.1) holds with $\mathbf{F}(T) = \mathcal{C}([0;T], H^s) \cap L_t^4 L^\infty$

and $\mathbf{N}(T) = L_T^{4/3}L^1$. To deal with (1.2.2), we use Hölder's inequality to get

$$\left|\left||u|^2u\right|\right|_{L_T^{4/3}L^1} \lesssim T^{1/2}\,||u||_{L_T^\infty L^2}^2\,||u||_{L_T^4L^\infty} \lesssim T^{1/2}\,||u||_{\mathbf{F}(T)}^3$$

This allows us to perform a fixed point argument in a closed ball of $\mathbf{F}(T)$ just as in subsection 1.2.1 to get existence, and uniqueness follows from (1.2.6) with the Sobolev embedding $L^\infty([0;T],H^{(1/2)+}) \hookrightarrow L^1([0;T],L^\infty)$ being replaced by the Strichartz estimate. This provides semilinear well-posedness for this equation, even global thanks to the conservation of the $L^2$ norm [Tsu87].

As we can see, in the argument above the boundedness of $\mathfrak{D}\circ\mathcal{N}$ relies only on the Strichartz estimate, so can one expect to tackle problems with a derivative loss ?

Let us look now at the Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0 \tag{1.2.14}$$

In view of the energy estimate (1.2.8), the key quantity to bound is $||\partial_x u||_{L_T^1L^\infty}$. In order to do so, we can look for a replacement of (1.2.12). The Strichartz estimate reads here

$$\left|\left|\mathrm{e}^{-t\partial_x^3}|D|^{1/4}u_0\right|\right|_{L_t^4L^\infty} \lesssim ||u_0||_{L^2} \tag{1.2.15}$$

where $|D|^s$ is the Fourier multiplier with symbol $|\xi|^s$. As we can see, (1.2.15) recovers only $1/4$ of the derivative, so we cannot perform a contraction argument directly in the space $\mathbf{X}_T = \mathcal{C}([0;T],H^s)\cap\{\partial_x u \in L_T^4L^\infty\}$. More precisely, using the Strichartz estimate (1.2.15), we have the linear estimate

$$||\mathcal{X}(u)||_{\mathbf{X}_T} \lesssim ||u_0||_{H^s} + T^{1/2}\,||u\partial_x u||_{L_T^2H^s}$$

which holds for $s>3/4$ and $T\in]0;1]$. For the nonlinear estimate (1.2.2), we use the commutator estimate (1.2.9) and the fact that $H^s(\mathbb{R})$ is still a algebra at this level of regularity, to get the bound

$$T^{1/2}\left\{||u||_{\mathbf{X}_T}^2 + ||u\cdot|D|^s\partial_x u||_{L_{T,x}^2}\right\}$$

In order to close the nonlinear estimate, we thus see that we need to control $||u\cdot\partial_x|D|^s u||_{L_{T,x}^2}$. Hence we finally take for the abstract contraction principle the spaces

$$\mathbf{F}(T) := \mathbf{X}_T \cap \left\{|D|^s\partial_x u \in L_x^\infty L_T^2\right\}\cap L_x^2L_T^\infty \text{ and } \mathbf{N}(T) = L_T^2H^s$$

Therefore we are left with proving the linear estimate for the contribution of the two extra norms we added to $\mathbf{X}_T$. This is performed thanks to the smoothing estimate

$$\left|\left|\partial_x\mathrm{e}^{-t\partial_x^3}u_0\right|\right|_{L_x^\infty L_t^2} \lesssim ||u_0||_{L^2} \tag{1.2.16}$$

along with the maximal function estimate

$$\left|\left|\mathrm{e}^{-t\partial_x^3}u_0\right|\right|_{L_x^2L_T^\infty} \lesssim ||u_0||_{H^s} \tag{1.2.17}$$

which holds for any $s>3/4$ and $T\in]0;1]$. Thus existence follows again from the contraction principle in a closed ball of $\mathbf{F}(T)$ and uniqueness in the whole class $\mathcal{C}([0;T],H^s)\cap\{\partial_x u\in L_T^4L^\infty\}$ follows from the energy estimate above, hence the local well-posedness in $H^s$, $s>3/4$ [KPV91].

As remarked in [KPV93b], this is as far as one can go with this method for this equation, since

$$||\partial_x S(t)u_0||_{L_T^1L^\infty} \lesssim ||u_0||_{H^s} \iff s \geqslant 3/4$$

Thus another approach is needed in order to get well-posedness at lower regularity.

### 1.2.4 The Fourier restriction norm method of Bourgain for semilinear equations

In the previous method, the abstract linear estimate (1.2.1) was a consequence of the Strichartz estimate (and its inhomogeneous version), and the nonlinear one followed from Hölder's inequality, whereas $\mathbf{F}(T)$ was the space associated with the Strichartz type estimates and $\mathbf{N}(T)$ its dual.

Now the previous method collapses when one moves to periodic equations. Indeed, the dispersion relation (1.2.11) clearly fails globally in time (since on a compact domain the decay of the $L^\infty$ norm contradicts the conservation of the $L^2$ norm), but even locally (see [BGT04, Remark 2.6]). Moreover, for nonperiodic equations, as remarked before we have pushed the precedent method to its limits. Thus another choice of spaces $(\mathbf{F}, \mathbf{N})$ must be made.

The groundbreaking idea of Bourgain [Bou93a] was to introduce the spaces $X^{s,b}$ (see also [RR82] in the context of nonlinear wave equation), defined through the norm

$$||u||_{X^{s,b}} := ||S(-t)u(t)||_{H^b_t H^s} = \left|\left|\langle i(\partial_t - \mathcal{L})\rangle^b \langle i\nabla\rangle^s u\right|\right|_{L^2} \tag{1.2.18}$$

We can already make a few comments : first, we see that in the time variable, $u$ is measured in a Sobolev type space (instead of a Lebesgue space as before) to take into account the gain of regularity in time within the Duhamel operator. Second, again from the definition these spaces are precisely tailored to measure the linear flow. In particular, the linear estimate

$$||\psi_T u||_{\mathbf{F}} \lesssim ||u_0||_{H^s} + ||\psi_T\left(\partial_t - \mathcal{L}\right)u||_{\mathbf{N}} \tag{1.2.19}$$

with $\mathbf{F} = X^{s,b}$, $\mathbf{N} = X^{s,b-1}$ and $\psi_T$ is a smooth characteristic function of $[0;T]$, now reduces to a general time variable estimate

$$\begin{cases} ||\psi_T u_0||_{H^b_t H^s} \lesssim ||u_0||_{H^s} \\ \left|\left|\psi_T \int_0^t g(t')\mathrm{d}t'\right|\right|_{H^b} \lesssim ||g||_{H^{b-1}} \end{cases} \tag{1.2.20}$$

for any $b > 1/2$, and no Strichartz estimate is needed at this stage. And third, the functions in $X^{s,b}$ are defined on the entire space $\mathbb{R} \times \Omega$, so in order to have $\mathbf{F} \subset \mathcal{C}([0;T], H^s)$ one needs to truncate in time and taking the norm

$$||u||_{\mathbf{F}(T)} = \inf\{||\widetilde{u}||_{X^{s,b}}\,,\ \widetilde{u} \equiv u \text{ on } [0;T]\}.$$

This is possible since $X^{s,b}$ spaces are stable by multiplication by smooth functions in the time variable :

$$||\psi_T u||_{X^{s,b'}} \lesssim T^{b-b'}\,||u||_{X^{s,b}} \tag{1.2.21}$$

for any $b' \leqslant b$.

Thus in order to get a contraction property all the difficulty is concentrated on the nonlinear term. In the case of a power nonlinearity $\mathcal{N}(u) = \partial_x f(u)$ with $f(u) = u^p$, we are thus left with proving a *multilinear estimate* of the form

$$\left|\left|\partial_x \prod_{i=1}^p u_i\right|\right|_{X^{s,b-1}} \lesssim \prod_{i=1}^p ||u_i||_{X^{s,b'}} \tag{1.2.22}$$

with $b' < b$.

To prove (1.2.22), one can now use the Strichartz and local smoothing estimates : as explained in [Gin96, Tao06], one has the general estimate

$$||f||_Y \lesssim ||f||_{X^{0,(1/2)+}} \tag{1.2.23}$$

for any space-time functional space $Y$ stable by multiplication by bounded time dependent functions. In particular, one can take for $Y$ the $L_t^p L_x^q$ type space involved in the Strichartz type estimates.

Let us come back to the cubic NLS (1.2.10) and illustrate how the Strichartz estimate (1.2.12) can be injected in the $X^{s,b}$ framework through (1.2.23) : interpolating between the conservation of the $L^2$ norm and (1.2.12), we obtain first the $L^6$ estimate

$$\left|\left|\mathrm{e}^{-it\partial_x^2} u_0\right|\right|_{L^6_{t,x}} \lesssim ||u_0||_{L^2}$$

Applying the principle (1.2.23), this provides the bound

$$||u||_{L^6_{t,x}} \lesssim ||u||_{X^{0,(1/2)+}}$$

and another interpolation with the obvious equality $||f||_{L^2_{t,x}} = ||f||_{X^{0,0}}$ leads to

$$||f||_{L^4_{t,x}} \lesssim ||f||_{X^{0,(3/8)+}} \tag{1.2.24}$$

This allows to close the fixed point argument in $X_T^{0,(1/2)+}$, since the trilinear estimate is a consequence of

$$\begin{aligned} \left|\left||u|^2 u\right|\right|_{X^{0,(-1/2)+}} \lesssim T^{(1/8)-} \left|\left||u|^2 u\right|\right|_{X^{0,-(3/8+)}} \lesssim T^{(1/8)-} \left|\left||u|^2 u\right|\right|_{L^{4/3}_{t,x}} \\ \lesssim T^{(1/8)-} ||u||^3_{L^4_{t,x}} \lesssim T^{(1/8)-} ||u||^3_{X^{0,(3/8)+}} \end{aligned}$$

where the first and second estimates are the dual versions of (1.2.21) and (1.2.24). This leads to global semilinear well-posedness in $L^2(\mathbb{R})$, which was already proved by the previous method [Tsu87]. However, to estimate the nonlinear term we resorted to (1.2.24), and since $X^{s,b}$ is precisely the space where linear solutions live, one can prove the same estimate for periodic solutions directly in these spaces without using the intermediate steps. In the case of the periodic cubic NLS, Bourgain [Bou93a] proved that (1.2.24) holds (and even the endpoint case $b = 3/8$) by a direct estimate on the Fourier series. This allowed him to get the global semilinear well-posedness in $L^2(\mathbb{T})$ through the same argument as above.

In the case of a derivative nonlinearity, the same procedure can be applied : if we look again at the KdV equation (1.2.14), interpolating between the Strichartz estimate (1.2.15) and the trivial estimate $\left|\left|\mathrm{e}^{-t\partial_x^3} u_0\right|\right|_{L_t^\infty L^2} = ||u_0||_{L^2}$ and using a Sobolev inequality provides the estimate

$$\left|\left|\mathrm{e}^{-t\partial_x^3} u_0\right|\right|_{L^8_{t,x}} \lesssim \left|\left||D|^{1/8}\mathrm{e}^{-t\partial_x^3} u_0\right|\right|_{L^8_t L^4_x} \lesssim ||u_0||_{L^2}$$

which gives, through the principle (1.2.23), the estimate

$$||f||_{L^8_{t,x}} \lesssim ||f||_{X^{0,(1/2)+}}$$

Interpolating again with the obvious equality $||f||_{L^2_{t,x}} = ||f||_{X^{0,0}}$, we obtain the bound

$$||f||_{L^4_{t,x}} \lesssim ||f||_{X^{0,(1/3)+}} \tag{1.2.25}$$

On the other hand, we can also interpolate between (1.2.16) and (1.2.17) to get for $T \in ]0;1]$

$$\left|\left|\mathrm{e}^{-t\partial_x^3} u_0\right|\right|_{L^4_{T,x}} \lesssim \left|\left||D|^{-(1/8)+} u_0\right|\right|_{L^2}$$

Thus, using again principle (1.2.23) and interpolating with the previous bound, we finally get

$$||f||_{L^4_{T,x}} \lesssim ||f||_{X^{0-,(1/2)-}} \tag{1.2.26}$$

We can finally use this bound to prove the bilinear estimate (1.2.22) : for $u, v \in X^{0,b'}$, $1/2 < b' < b$, we can define $U(\tau,\xi) := \left\langle \tau - \xi^3 \right\rangle^{b'} \mathcal{F}_{t,x}(u)(\tau,\xi)$ such that $||u||_{X^{0,b'}} = ||U||_{L^2}$, and similarly for $v$. Then by duality (1.2.22) can be rewriten as

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} &\frac{(\xi_1+\xi_2)}{\left\langle \tau_1+\tau_2-\xi_1^3-\xi_2^3\right\rangle^{1-b}\left\langle \tau_1-\xi_1^3\right\rangle^{b'}\left\langle \tau_2-\xi_2^3\right\rangle^{b'}} \\ &\cdot U(\tau_1,\xi_1)V(\tau_2,\xi_2)w(\tau_1+\tau_2,\xi_1+\xi_2)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\xi_1\mathrm{d}\xi_2 \lesssim ||U||_{L^2}\,||V||_{L^2}\,||w||_{L^2}\end{aligned} \tag{1.2.27}$$

As before, there are two main difficulties to deal with : the presence of a derivative in the nonlinearity, and the power nonlinearity itself. For the first one, a key ingredient in the proof of Bourgain is that the term $\left\langle \tau_1+\tau_2-\xi_1^3-\xi_2^3\right\rangle^{1-b}$ (which expresses a gain of temporal regularity) can be written as

$$\left\langle \tau_1+\tau_2-(\xi_1+\xi_2)^3+\Omega(\xi_1,\xi_2)\right\rangle^{1-b}$$

where the *resonant function* is defined as

$$\Omega(\xi_1,\xi_2) = (\xi_1+\xi_2)^3 - \xi_1^3 - \xi_2^3 = 3\xi_1\xi_2(\xi_1+\xi_2) \tag{1.2.28}$$

This algebraic identity implies that

$$\max\left\{\left\langle \tau_1+\tau_2-(\xi_1+\xi_2)^3\right\rangle, \left\langle \tau_1-\xi_1^3\right\rangle, \left\langle \tau_2-\xi_2^3\right\rangle\right\} \gtrsim |\xi_1\xi_2(\xi_1+\xi_2)| \tag{1.2.29}$$

To end the argument, we divide the domain of integration in subdomains depending on which term dominates in the fraction above and on the symmetries in $\tau_i$ and $\xi_i$. For example, for the case $\xi_2 \geqslant \xi_1 \geqslant 1$ and $\max\{...\} = \left\langle \tau_1+\tau_2-(\xi_1+\xi_2)^3\right\rangle$, then the contribution of this region in the left-hand side of (1.2.27) can be bounded by

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \xi_2^{2b-1}\left\langle \tau_1-\xi_1^3\right\rangle^{-b'}\left\langle \tau_2-\xi_2^3\right\rangle^{-b'} |U(\tau_1,\xi_1)V(\tau_2,\xi_2)w(\tau_1+\tau_2,\xi_1+\xi_2)|\,\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\xi_1\mathrm{d}\xi_2$$

Using Cauchy-Schwarz, Plancherel and Hölder, we estimate this last term with

$$\left|\left|\mathcal{F}^{-1}\left\{\left\langle \tau-\xi^3\right\rangle^{-b'}|U|\right\}\right|\right|_{L^4}\left|\left|\mathcal{F}^{-1}\left\{\langle\xi\rangle^{2b-1}\left\langle \tau-\xi^3\right\rangle^{-b'}|V|\right\}\right|\right|_{L^4}||w||_{L^2} \lesssim ||U||_{L^2}\,||V||_{L^2}\,||w||_{L^2}$$

where the last inequality follows from (1.2.26). This leads global semilinear well-posedness in $L^2(\mathbb{R})$. Again, Bourgain proved directly that (1.2.25) holds also for periodic functions (with $b' = 1/3$), hence the same result on $L^2(\mathbb{T})$ [Bou93a]. Actually, further interpolations between (1.2.15)-(1.2.16)-(1.2.17) allow to take $s > -5/8$ on $\mathbb{R}$ [KPV93a]).

Two further improvements can be made in the setting of $X^{s,b}$ spaces : the first one is that, when (1.2.24) does not hold, one can look for a version of this estimate with loss of derivative. That is, we look for estimates of the form

$$||P_N u||_{L^q_{t,x}} \lesssim N^\alpha\,||P_N u||_{X^{0,b}} \tag{1.2.30}$$

where $P_N$ is a projector on the set of frequencies $|\xi| \sim N$. For example, again for the cubic NLS equation on $\mathbb{T}$, Bourgain proved that (1.2.30) holds with $q = 6$ and $\alpha = 0+$. These frequency localized estimates allow us to cut the integral in (1.2.27) in dyadic pieces and then to evaluate the contributions of these dyadic regions depending on the relation between the dyadic numbers $N_i$ and to use (1.2.30) for each piece. This remark leads to the second and most significant improvement : instead of estimating each frequency piece of one linear solution as in (1.2.30), one can bound directly the interaction between two (or more) linear solutions. For example, one has the bilinear improvement of (1.2.24)

$$||P_{N_1}u_1 \cdot P_{N_2}u_2||_{L^2_{t,x}} \lesssim (N_1 \vee N_2)^{-1/2}\,||u_1||_{X^{0,(1/2)+}}\,||u_2||_{X^{0,(1/2)+}} \tag{1.2.31}$$

which holds in the regime $(N_1 \vee N_2) \gg (N_1 \wedge N_2)$. Tao [Tao01] made a systematic study of estimates of type (1.2.31). Coming back to the KdV equation, such a strategy was successfully carried out by Kenig, Ponce and Vega [KPV96] who proved local well-posedness in both $H^s(\mathbb{R})$, $s > -3/4$ and $H^s(\mathbb{T})$, $s > -1/2$, and that this regularity is critical in the sense of semilinear well-posedness. Actually, the proof of this result relies on a direct computation of the bilinear interactions within the integral (1.2.27) without resorting to dyadic decompositions, yet Tao [Tao01] proved that both approaches are equivalent and the former allows to unify the treatment of both periodic and nonperiodic multilinear estimates.

We can cite another result in the same spirit as above : Bourgain [Bou93b] proved semilinear global well-posedness of the KP-II equation

$$\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{1.2.32}$$

in both $L^2(\mathbb{T}^2)$ and $L^2(\mathbb{R}^2)$. His argument relies on the two features that we have encountered : a bilinear Strichartz type estimate for frequency localized functions, and a smoothing effect due to the algebraic identity

$$\begin{aligned}\big|(m_1+m_2)^3 - (n_1+n_2)^2/(m_1+m_2) - m_1^3 + n_1^2/m_1 - m_2^3 + n_2^3/m_2\big| \\ = \left|\frac{m_1 m_2}{m_1+m_2}\left\{3(m_1+m_2)^2 + \left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right)^2\right\}\right| \\ \gtrsim |m_1 m_2(m_1+m_2)|\end{aligned} \tag{1.2.33}$$

which is similar to the one for the KdV equation (1.2.29) and allows to recover the derivative in the nonlinearity.

Let us resume what we have obtained so far : in the case of a pure power nonlinearity, this method allows to have a unified treatment of the periodic and nonperiodic cases, and for this latter the bilinear refinement allows to improve on the results given by the previous method. In presence of a derivative, we have seen that for both the periodic and nonperiodic cases, this improves on the previous method under the condition that the resonant set defined by

$$\begin{cases}\sum_{i=1}^p \xi_i = 0\\ \sum_{i=1}^p \omega(\xi_i) = 0\end{cases} \tag{1.2.34}$$

where $\omega$ is the symbol of $\mathcal{L}$, is simple enough. And in both these settings, the estimates obtained close the fixed point argument and yield semilinear well-posedness. In the light of these results, is it possible to adapt the method to equations presenting a larger set of resonant frequencies ?

Molinet, Saut and Tzvetkov [MST02b] have proved that in the case of the KP-I equation, the resonant set contains a non trivial low-high interaction (change the plus sign in a minus sign in the second line of (1.2.33) above) which causes the bilinear estimate to fail in any Sobolev type space. This is even more pathological : it prevents the flow map from being smooth (even $\mathcal{C}^2$). This implies that *any* fixed point method is destined to fail, and in particular one has to find a general approach different from the one explained at the beginning of this subsection. We will see how to improve on the standard energy method of subsection 1.2.2 in this case.

### 1.2.5 A refined energy method for quasilinear equations

As mentioned above, the contraction principle must be forsaken when dealing with quasilinear equations. Heuristically, the lack of regularity of the flow map arises when the nonlinear effect supplants the linear evolution in fixed time. Let us explain this through the KP-I equation

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0,\ (t,x,y)\in\mathbb{R}\times\mathbb{T}^2 \tag{1.2.35}$$

We have already seen that, compared to the KP-II equation, this equation enjoys a large set of resonant frequencies which prevents the smoothing effect allowing to recover the derivative in the nonlinearity. Let us see why it also prevents any fixed point method to be successfull. This is better illustrated in the periodic setting : we start from a (complex-valued) datum

$$u_0(x,y) = \mathrm{e}^{ix} + n^{-s}\mathrm{e}^{i(nx+\alpha(n)y)}$$

where $\alpha(n)$ is chosen such that $(1,0,n,\alpha(n))$ lies in the resonant set, and $u_0$ is normalized such that $||u_0||_{H^{s,0}} \sim 1$, where $H^{s,0}$ refers to the anisotropic Sobolev space with norm

$$||u_0||_{H^{s_1,s_2}} := ||\langle D_x\rangle^{s_1}\langle D_y\rangle^{s_2}u_0||_{L^2} = ||\langle n\rangle^{s_1}\langle k\rangle^{s_2}\widehat{u_0}(n,k)||_{\ell^2}$$

The first Picard iteration yields the linear solution

$$u_1(t,x,y) = \mathrm{e}^{-t(\partial_x^3-\partial_x^{-1}\partial_y^2)}u_0 = \mathrm{e}^{i\varphi_1} + n^{-s}\mathrm{e}^{i\varphi_n}$$

where $\varphi_1(t,x,y) = x+t$ and $\varphi_n(t,x,y) = nx+\alpha(n)y+t\alpha(n)^2/n$ are the phase functions of the free solutions. Our choice of frequencies means that

$$\varphi_1+\varphi_n = (n+1)x+\alpha(n)y+t\alpha(n)^2/(n+1) =: \varphi_{n+1}$$

is also the phase function of a free solution. Thus the second iterate can be computed as

$$u_2(t,x,y) = u_1(t,x,y) - \int_0^t \mathrm{e}^{-(t-t')(\partial_x^3-\partial_x^{-1}\partial_y^2)}\left\{\mathrm{e}^{i\varphi_1(t')}\cdot\partial_x n^{-s}\mathrm{e}^{i\varphi_n(t')}\right\}\mathrm{d}t' + r(t,x,y)$$

with $||r(t)||_{H^{s,0}} = O(1)$ as $n\to+\infty$. Now, since we chose resonant frequencies, the dominant nonlinear term in $u_2$ becomes

$$\int_0^t \mathrm{e}^{-(t-t')(\partial_x^3-\partial_x^{-1}\partial_y^2)}n^{1-s}\mathrm{e}^{i\varphi_{n+1}(t')}\mathrm{d}t' = tn^{1-s}\mathrm{e}^{i\varphi_{n+1}(t)}$$

Finally, we get $||u_2(t)||_{H^{s,0}} \sim |t|n + O(1)$. So, in order for $u_2$ to behave like the linear flow (which preserves the $H^{s,0}$ norm), we must have $|t|\sim n^{-1}$. This suggests that if one wants to get a unique solution in a fixed interval of time from a data $u_0$ then one has to control each frequency piece localized around $N$ on time intervals of size $N^{-1}$, and then sum up these estimates. This idea was first used by Koch and Tzvetkov [KT05] in the context of the Benjamin-Ono equation. Another way to state it is that even if, in the standard energy method, the regularity $s$ for which the estimate $||\partial_x u||_{L^1_T L^\infty} \lesssim ||u||_{L^\infty_T H^s}$ holds for any time $T > 0$ is limited, it still holds at lower regularity on time intervals of size $N^{-\alpha}$, $\alpha > 0$, when $u$ is replaced by $P_N u$. Of course, the greater $\alpha$ is, the lower $s$ can be taken, but then the number of such time intervals needed to cover $[0; T]$ for a fixed $T > 0$ becomes larger, thus there is a balance to find, with an optimal choice for $\alpha$. Let us illustrate this on the Benjamin-Ono equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0 \tag{1.2.36}$$

where $\mathcal{H}$ is the Hilbert transform, defined as the Fourier multiplier

$$\widehat{\mathcal{H}u}(\xi) := i\text{sign}(\xi)\widehat{u}(\xi)$$

Integrating on $[0; T]$ the standard energy estimate, we have

$$||u||^2_{L^\infty_T H^s} \lesssim ||u_0||^2_{H^s} + ||\partial_x u||_{L^1_T L^\infty} ||u||^2_{L^\infty_T H^s}$$

so that once again we seek to control $||\partial_x u||_{L^1_T L^\infty}$. Equation (1.2.36) admits the same Strichartz estimate as the nonlinear Schrödinger equation (1.2.12). In particular, for frequency localized solutions $P_N u$, we can use the Duhamel formula on time intervals $I = [a; b]$ to get the bound

$$||\partial_x P_N u||_{L^1(I, L^\infty)} \lesssim |I|^{3/4} N \left\{ ||P_N u(a)||_{L^2} + N \left|\left|P_N(u^2)\right|\right|_{L^1(I, L^2)} \right\}$$

To get an estimate on a fixed time interval $[0; T]$, we can cover it with $\approx T/|I|$ time intervals of length $|I|$ as above, and then sum their contribution in the left-hand side, thus getting the bound

$$||\partial_x P_N u||_{L^1_T L^\infty} \lesssim T|I|^{-1/4} N ||P_N u||_{L^\infty_T L^2} + |I|^{3/4} N^2 \left|\left|P_N(u^2)\right|\right|_{L^1_T L^2}$$

Finally, in order to have both the linear and nonlinear terms at the same regularity, we must take $|I| = N^{-1}$. So we readily see that the size of the time intervals $I$ recovers 3/4 of the derivative in the nonlinearity (this is possible because there is an integral in time in front of the nonlinearity, thus we can sum the contributions of the small time intervals in $L^1_T$ without gaining a power of $|I|$). The nonlinear term is then estimated classically since at this level of regularity ($s > 5/4$) $H^s(\mathbb{R})$ is still an algebra. Thus after summing the contribution of the spectral projections $P_N u$, we obtain an a priori bound on solutions and we can finish the compactness argument as in Kato's theory, to obtain local well-posedness in $H^s(\mathbb{R})$, $s > 5/4$ [KT03].

We can summarize the above procedure to get the a priori bounds needed for the compactness argument as

$$||u||_{\mathbf{F}(T)} \lesssim ||u||_{\mathbf{B}(T)} + ||(\partial_t - \mathcal{L})u||_{\mathbf{N}(T)} \text{ (a linear estimate)} \tag{1.2.37}$$

$$\left|\left|\partial_x(u^2)\right|\right|_{\mathbf{N}(T)} \lesssim ||u||^2_{\mathbf{F}(T)} \text{ (a nonlinear estimate)} \tag{1.2.38}$$

$$||u||^2_{\mathbf{B}(T)} \lesssim ||u_0||^2_{H^s} + ||u||_{\mathbf{F}(T)} ||u||^2_{\mathbf{B}(T)} \text{ (an energy estimate)} \tag{1.2.39}$$

where $\mathbf{F}(T) = \mathcal{C}([0; T], H^s) \cap \left\{\partial_x u \in L^1_T L^\infty\right\}$, $\mathbf{N}(T) = L^1_T H^{s-1}$ and $\mathbf{B}(T) = L^\infty_T H^s$.

In particular, compared to (1.2.1)-(1.2.2) (which cannot hold for any choice of spaces $\mathbf{F}$ and $\mathbf{N}$ due to the quasilinear nature of the equation), we see that in the linear estimate (1.2.37) the first term in the right-hand side (corresponding to the free solution) introduces an auxiliary space $\mathbf{B}(T)$ instead of the data $||u_0||_{H^s}$, which in turn requires a third estimate (1.2.39) on this term to close the a priori bound. This is because to derive the linear estimate, one applies the Duhamel formula on each small time subinterval where the data is actually the value of the solution at the boundary of the interval, thus when one recombines these estimates one has to control all these boundary values instead of just the one at $t = 0$.

As for the fixed point methods, we see that the choice of the spaces $\mathbf{F}(T)$, $\mathbf{N}(T)$ and $\mathbf{B}(T)$ is crucial. The next section explains how to tailor them to fit a given equation.

### 1.2.6 The small time Fourier restriction norm method

As we have seen before, the major benefit of resorting to $X^{s,b}$ spaces over $L_t^pL_x^q$ type spaces is to get a bilinear improvement over the Strichartz type estimates. In the refined energy method above, we used the latter framework, thus can we adapt the former in the context of frequency dependent small time spaces ?

This procedure has been carried out by Ionescu, Kenig and Tataru [IKT08] in the context of the KP-I equation on $\mathbb{R}^2$. The general idea is again to find some functional spaces $\mathbf{F}(T) \subset \mathcal{C}([0;T],H^s)$, $\mathbf{N}(T)$ and $\mathbf{B}(T)$ in which (1.2.37)-(1.2.38)-(1.2.39) hold.

First, let us fix a dyadic number $N$ and project the functions $u$ on the frequency region $|\xi| \sim N$. Then, the computation at the beginning of the previous subsection suggests that the resonant low-high interaction should be controllable on time intervals of size $N^{-\alpha}$ for some $\alpha > 0$.

For fixed $N$, we thus consider a time interval $I_N = [a;b]$ of size $N^{-\alpha}$, and we estimate $\psi_{I_N}P_Nu$ in $X^{s,1/2}$ (here we use a Besov version of the Bourgain space to recover the limit case $b = 1/2$). As for the standard $X^{s,b}$ spaces, we then have the linear estimate

$$||\psi_{I_N}P_Nu||_{X^{s,1/2}} \lesssim ||P_Nu(a)||_{H^s} + ||\psi_{I_N}P_N(\partial_t - \mathcal{L})u||_{X^{s,-1/2}} \tag{1.2.40}$$

The first term in the right-hand side of (1.2.40) is estimated with $||P_Nu||_{L^\infty(I_N,H^s)}$. This implies that we then need to control all the contributions of the small intervals in an $\ell^\infty$ manner : otherwise, any sum on the $I_N$ of the $L^\infty(I_N)$ norms above would lose a factor $TN^\alpha$ corresponding to the number of such intervals needed to cover $[0;T]$, and this loss of derivative at this stage would not be recovered in the subsequent estimates. Thus the space $\mathbf{F}$ (respectively $\mathbf{B}$, $\mathbf{N}$) is constructed as follow : first, localize in frequency and time and measure in the $X^{s,1/2}$ space (respectively $L_t^\infty H^s$, $X^{s,-1/2}$), then take the sup on the small intervals and then sum up in frequencies in a Besov way.

As for the second term, we have to prove a small-time analogue of the dyadic bilinear Strichartz type estimate (1.2.31) : after splitting $u_1$ and $u_2$ in dyadic pieces, we then have to control $||\psi_{I_N}P_N\partial_x(u_{N_1}\cdot u_{N_2})||_{X^{s,-1/2}}$. Now, we see how this small time truncation helps with the resonant low-high interaction : if $N_1 \ll N_2 \sim N$ then we can move the time localization to both $u_{N_1}$ and $u_{N_2}$ (for the former this is allowed since, again, we require a uniform control on all the small time intervals). This implies that, even if we are not in the KP-II regime (1.2.33), according to the uncertainty principle the modulations $\langle\tau_i - \omega(\xi_i)\rangle$ are still bounded from below by $N_i$, thus providing an extra smoothing effect (compared to the standard bilinear estimate) which helps to recover the derivative. Of course, the key point of the analysis is to establish the estimate

$$\left|\left|\psi_{I_N}P_N\partial_x\mathfrak{R}(\psi_{I_{N_1}}u_{N_1}\cdot\psi_{I_{N_2}}u_{N_2})\right|\right|_{X^{s,-1/2}} \lesssim \left|\left|\psi_{I_{N_1}}u_{N_1}\right|\right|_{X^{s,1/2}}\left|\left|\psi_{I_{N_2}}u_{N_2}\right|\right|_{X^{s,1/2}} \tag{1.2.41}$$

where $\mathfrak{R}$ is the projection on the set of frequencies where the KP-II smoothing relation fails. The remaining non resonant interaction is estimated with the same arguments as in the standard Bourgain method, thus proving (1.2.38).

Finally, we see on the second term of (1.2.40) that we need the energy estimate to close the a priori bound. It is derived similarly to the standard energy estimate, but after the integration by parts and commutator estimate, we use again a dual version of the dyadic bilinear estimate above (and the corresponding one for the non resonant interaction), thus leading to

$$||u||^2_{\mathbf{B}(T)} \lesssim ||u_0||^2_{H^s} + ||u||^3_{\mathbf{F}(T)} \tag{1.2.42}$$

Combining (1.2.37)-(1.2.38)-(1.2.42) leads to an a priori bound on $||u||_{\mathbf{F(T)}}$ which allows us to end the compactness argument as in the previous method. Let us finally outline that we have to use again the Bona-Smith argument to get only the continuity property of the flow map.

# Chapter 2

# Statement of the results

This chapter is an english version of chapter 0 section 0.3.

## 2.1 The Cauchy problem for the KP-I equation

This section is devoted to the study of the KP-I equation

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{2.1.1}$$

This equation is described by the general formalism of the previous chapter, with $\mathcal{L} = -\partial_x^3 + \partial_x^{-1}\partial_y^2$, and the operator $\partial_x^{-1}$ being defined as the (singular) Fourier multiplier with symbol $\frac{1}{i\xi}$, and here the nonlinearity is $\mathcal{N}(u) = \partial_x f(u) = \partial_x(-u^2/2)$.

### 2.1.1 Previous results on the Cauchy problem

The Cauchy problem for this equation has been studied extensively for a few decades. The first results were due to several authors [Uka89, IMS92, Sau93, IMS95, IN98] by mean of Kato's theory, to get local well-posedness of (2.1.1) (and some more general power nonlinearity) in a subspace of $H^s(\Omega)$ adapted to the singular symbol of $\mathcal{L}$, where $s > 1 + d/2 = 2$ and $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{T}^2$. As we have already seen in the previous chapter, this general method does not take into account the precise form of the operator $\partial_x^3 - \partial_x^{-1}\partial_y^2$, and in particular it applies equally to the KP-II equation for which the linear operator becomes $\partial_x^3 + \partial_x^{-1}\partial_y^2$. This former equation has then been studied by Bourgain [Bou93b] who obtained global semilinear well-posedness in $L^2(\Omega)$ for both $\Omega = \mathbb{R}^2$ and $\Omega = \mathbb{T}^2$ by using a contraction principle in $X^{s,b}$ spaces. As we have already encountered, the proof of the bilinear estimate heavily relies on the sign within the symbol $\omega_{II}(m,n) := m^3 - n^2/m$ for the linear operator in the KP-II equation, and more precisely on the algebraic relation

$$\begin{aligned}\omega_{II}(m_1+m_2, n_1+n_2) - \omega_{II}(m_1,n_1) - \omega_{II}(m_2,n_2)\\ = \frac{m_1m_2}{m_1+m_2}\left\{3(m_1+m_2)^2 + \left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right)^2\right\}\end{aligned} \tag{2.1.2}$$

which shows that the KP-II equation only has trivial resonances, when one frequency vanishes, which can be removed by the Galilean transform (1.1.16). The general strategy of subsection 1.2.4

was improved in a series of papers [TT01, IM01, Had08] and was at its height with the result of Hadac, Herr and Koch [HHK09] who proved (small data) semilinear global well-posedness in the scale critical space $\dot{H}^{-1/2,0}(\mathbb{R}^2)$. The problem was shown to be also semilinearly globally well-posed in $L^2(\mathbb{R}\times\mathbb{T})$ [MST11].

Concerning the KP-I equation (2.1.1), the problem behaves differently : since the symbol is now $\omega_I(m,n) = m^3 - n^2/m$ then in (2.1.2) there is a minus sign instead of a plus sign between the positive terms. This means that there is a large set of resonant frequencies, which causes the bilinear estimate to fail on $\mathbb{R}^2$ [MST02b], and even prevents the flow map from being of class $\mathcal{C}^2$, thus condemning any attempt via a contraction principle. Moreover, Koch and Tzvetkov [KT08] have proved that the problem is indeed semilinearly ill-posed in the sense of definition 1.1.3, by exhibiting two families of solutions which are initially arbitrarily close but become distant as soon as time evolves.

By using the refined energy method of subsection 1.2.5, Kenig [Ken04] and then Ionescu and Kenig [IK07] got global well-posedness in the "second energy space"

$$\mathbf{Z}^2(\Omega) := \left\{u_0\in L^2,\ \partial_x^2 u_0\in L^2,\ \partial_x^{-2}\partial_y^2 u_0 \in L^2\right\} \tag{2.1.3}$$

where $\Omega = \mathbb{R}^2$ and $\Omega\in\left\{\mathbb{R}\times\mathbb{T},\mathbb{T}^2\right\}$ respectively. Indeed, the space (2.1.3) is related to another conserved quantity : besides the Hamiltonian

$$\mathcal{H}(u) = \frac{1}{2}\left|\left|\partial_x u\right|\right|_{L^2}^2 + \frac{1}{2}\left|\left|\partial_x^{-1}\partial_y u\right|\right|_{L^2}^2 - \frac{1}{6}\int_\Omega u^3 \mathrm{d}x\mathrm{d}y \tag{2.1.4}$$

the functional

$$\begin{aligned}\mathcal{Z}(u) = \frac{3}{2}\left|\left|\partial_x^2 u\right|\right|_{L^2}^2 + \frac{5}{6}\left|\left|\partial_x^{-2}\partial_y^2 u\right|\right|_{L^2}^2 + 5\left|\left|\partial_y u\right|\right|_{L^2}^2 - \frac{5}{6}\int u^2(\partial_x^{-2}\partial_y^2 u)\\ - \frac{5}{6}\int u(\partial_x^{-1}\partial_y u)^2 + \frac{5}{4}\int u^2\partial_x^2 u + \frac{5}{24}\int u^4\end{aligned} \tag{2.1.5}$$

is also (at least formally) conserved under the flow. Actually (2.1.1) admits an infinite number of conserved quantities, yet it is not clear how to find a functional space where they make sense [MST02a]. The proof of [Ken04, IK07] relies on a small time Strichartz time estimate for frequency localized data

$$\left|\left|\mathrm{e}^{t\mathcal{L}}P_M^x P_N^y u_0\right|\right|_{L^2(|t|\lesssim M^{-1}))L^\infty_{xy}} \lesssim \Lambda_\Omega(M,N)\left|\left|u_0\right|\right|_{L^2} \tag{2.1.6}$$

where $P_M^x$ and $P_N^y$ are spectral projectors. The loss $\Lambda_\Omega(M,N)$ in the above estimate depends on the geometry of the spatial domain :

$$\Lambda_\Omega(M,N) \lesssim \begin{cases} M^{0+} \text{ if } \Omega = \mathbb{R}^2\\ (1\vee NM^{-2})^{1/2}M^{0+} \text{ if } \Omega = \mathbb{R}\times\mathbb{T}\\ (1\vee NM^{-2})^{1/2}M^{(3/8)+} \text{ if } \Omega = \mathbb{T}^2\end{cases} \tag{2.1.7}$$

Let us note that, when $\Omega = \mathbb{R}^2$, estimate (2.1.6) holds even for time intervals of size $O(1)$ and for nonlocalized data [Sau93].

(2.1.6) allows then to proceed as in subsection 1.2.5 and to control $\left|\left|\partial_x u\right|\right|_{L^1_T L^\infty_{xy}}$.

In order to reach the natural energy space

$$\mathbf{E}(\mathbb{R}^2) := \left\{u_0\in L^2, \partial_x u_0\in L^2, \partial_x^{-1}\partial_y u_0\in L^2\right\} \tag{2.1.8}$$

associated with the Hamiltonian, Ionescu, Kenig and Tataru [IKT08] then implemented the short time Fourier restriction norm method introduced in subsection 1.2.6, obtaining thereby global wellposedness in this space.

The key point in the proof of [IKT08] is a bilinear estimate for the resonant interaction :

$$\sup_{|I|\sim M^{-1}} ||\psi_I(t)\partial_x \mathfrak{R}(P_{M_1}u\cdot P_{M_2}v)||_{X^{0,-1/2}} \lesssim (M_1\wedge M_2\wedge M)^{-1/2} \sup_{|I_1|\sim M_1^{-1}} ||\psi_{I_1}(t)P_{M_1}u||_{X^{0,1/2}} \sup_{|I_2|\sim M_2^{-2}} ||\psi_{I_2}(t)P_{M_2}v||_{X^{0,1/2}} \quad (2.1.9)$$

where the supremum is taken over all time intervals $I\subset[0;T]$, $P_{M_i}$ is the projection on frequencies $|m|\sim M_i$ and $\mathfrak{R}$ is the projection of $P_{M_1}u\cdot P_{M_2}v$ on the set of frequencies satisfying

$$|\omega_I(m_1+m_2,n_1+n_2)-\omega_I(m_1,n_1)-\omega_I(m_2,n_2)|\lesssim |m_1m_2(m_1+m_2)|$$

i.e where the smoothing effect of KP-II fails.

In the case $\Omega=\mathbb{T}^2$, Zhang [Zha15] adapted the proof of [IKT08] and showed that (2.1.9) holds with a loss :

$$\sup_{|I|\sim M^{-1}} ||\psi_I(t)\partial_x \mathfrak{R}(P_{M_1}u\cdot P_{M_2}v)||_{X^{0,-1/2}} \lesssim \sup_{|I_1|\sim M_1^{-1}} ||\psi_{I_1}(t)P_{M_1}u||_{X^{0,1/2}} \sup_{|I_2|\sim M_2^{-2}} ||\psi_{I_2}(t)P_{M_2}v||_{X^{0,1/2}} \quad (2.1.10)$$

which restrains the range of regularity in $x$ and provides local well-posedness in a functional space strictly embedded in the energy space.

### 2.1.2 The Cauchy problem on a cylinder

The main result of this thesis is the following :

**Theorem 2.1.1**

*The Cauchy problem for (2.1.1) is globally well-posed in the energy space* $\mathbf{E}(\mathbb{R}\times\mathbb{T})$.

The proof of this theorem is the object of chapter 3. As for the proof of [IKT08], the key point is to show that (2.1.9) holds without loss in the case $\Omega=\mathbb{R}\times\mathbb{T}$. This requires to adapt the proof of (2.1.9) in the same manner as [Zha15] but taking advantage of the fact that the Fourier variable dual of $x$ is not an integer, which allows us to measure more accurately the small variations of the level lines of the resonant set. Let us underline that in the fully periodic case the counting measure is too rough to capture these small variations. For the non resonant contribution in the bilinear estimate, we also need to replace the standard Strichartz estimate [Sau93] which is no longer available : we then get the same estimate for small time and frequency localized data up to a small loss which requires a small modification of the short time Bourgain spaces of Ionescu, Kenig and Tataru. At last, we note that our uniqueness criterion is in the sense of defnition 1.1.1 and not only as the limit of smooth solutions.

### 2.1.3 Regarding the line soliton

Equation (2.1.1) is a two dimensional generalization of the KdV equation (1.2.14), which is known to admit special traveling waves solutions called *solitons*

$$u_c(t,x,y)=Q_c(x-ct) \text{ with } Q_c(x):=3c\cdot\cosh\left(\frac{\sqrt{c}}{2}x\right)^{-2} \quad (2.1.11)$$

Thus these solutions are also special solutions of the KP-I equation. However, they are not localized in the transverse direction (since they are independent of $y$) so they cannot belong to the energy space $\mathbf{E}(\mathbb{R}^2)$. This is what motivates the study of (2.1.1) on the spatial domain $\mathbb{R} \times \mathbb{T}$. The orbital stability of this class of solutions has been investigated by Rousset and Tzvetkov [RT12] : by using the spectral study of [APS97] and the general argument of [Ben72, GSS87] they obtained orbital stability of the family $(u_c)$ for small speeds $0 < c < 4/\sqrt{3}$ and under perturbations in $\mathbf{Z}^2(\mathbb{R} \times \mathbb{T})$ (2.1.3). Their argument actually relies only on the conservation of the Hamiltonian of (2.1.1) and the extra assumptions on the perturbations (namely, two derivatives in $x$ instead of one, and $\int_{\mathbb{R}} xu(x,y)\mathrm{d}x = 0$) were only technical since the only global well-posedness result known was that of Ionescu and Kenig [IK07] which used the second energy. Thus our global well-posedness result above shows that these assumptions can be removed, thus leading to

**Corollary 2.1.2**

*The line solitons $(u_c)$ are orbitally stable in $\mathbf{E}(\mathbb{R} \times \mathbb{T})$ for small speed $0 < c < 4/\sqrt{3}$.*

From [RT12], the family $(u_c)$ is known to be unstable for $c > 4/\sqrt{3}$. The critical case $c = 4/\sqrt{3}$ has been recently investigated by Yamazaki [Yam17]. His proof uses again the second energy to work with the corresponding flow on $\mathbf{Z}^2(\mathbb{R} \times \mathbb{T})$, yet it seems like the argument relies again only on the conservation of the Hamiltonian, and since the branch of the Zaitsev solutions (with which the branch of line solitons bifurcates at $c = 4/\sqrt{3}$) is also in the energy space, one can reasonably think that the corollary above extends to the critical speed.

Moreover, (2.1.1) possesses also solitons localized in both variables $x$ and $y$ [dBS97], called lumps. In a recent result, Liu and Wei [LW17] proved their non degeneracy and orbital stability in the energy space $\mathbf{E}(\mathbb{R}^2)$. The scheme of their proof is to study the non degeneracy of a family of $y$-periodic traveling waves $(Q_k)$, $k \in [0; 1/2]$ whose period goes to infinity as $k \to 0$ and which then converge to the lump. For $k \to 1/2$ then $Q_k$ converges to the above line soliton. Thus our theorem 2.1.1 provides a good setting to study the orbital stability of these traveling waves in the energy space $\mathbf{E}(\mathbb{R} \times \mathbb{T})$.

### 2.1.4 Concerning the regularity of the flow map

Molinet, Saut and Tzvetkov [MST02b] have proved that the flow map of (2.1.1) on $\mathbb{R}^2$ cannot be of class $\mathcal{C}^2$. This result relies on a resonant low-high frequency interaction in the nonlinearity. By exploiting the same resonances, we can adapt their result on $\mathbb{R} \times \mathbb{T}$ :

**Proposition 2.1.3**

*Let $(s_1, s_2) \in \mathbb{R}^2$. Then there exists no time $T > 0$ such that a flow map of (2.1.1) $\Phi_t : \mathbf{u}_0 \mapsto u(t)$, $t \in [-T; T]$ is $\mathcal{C}^2$-differentiable around 0 from*

$$H^{s_1,s_2}(\mathbb{R} \times \mathbb{T}) := \left\{u_0 \in L^2(\mathbb{R} \times \mathbb{T}),\ |D|_x^{s_1} u_0 \in L^2(\mathbb{R} \times \mathbb{T}),\ |D|_y^{s_2} u_0 \in L^2(\mathbb{R} \times \mathbb{T})\right\}$$

*to $H^{s_1,s_2}(\mathbb{R} \times \mathbb{T})$. The same result holds with $H^{s_1,s_2}$ being replaced by the energy space.*

This proposition motivates to drop the iteration methods (which provide a smooth flow map) and the use of a compactness argument. To get well-posedness in the sense of (1.1.1), we have to use Bona-Smith argument [BS75] to prove that the flow is still continuous. We then show that the problem is indeed semilinearly ill-posed in the sense of (1.1.3) by adapting the argument of Koch and Tzvetkov [KT08] :

**Proposition 2.1.4**

*The flow map for (2.1.1) is not uniformly continuous on the bounded sets of* $\mathcal{C}([-1;1], \mathbf{E}(\mathbb{R}\times\mathbb{T}))$.

## 2.2 About the fifth-order KP-I equation

Now we move to the study of the fifth-order KP-I equation

$$\partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{2.2.1}$$

### 2.2.1 A qualitative behaviour depending on the geometry of the domain

The Cauchy problem for (2.2.1) was previously studied by Saut and Tzvetkov [ST00, ST01]. By using the fixed point method of subsection 1.2.4, they obtained global semilinear well-posedness in both energy spaces $\mathbf{E}^2(\mathbb{R}^2)$ and $\mathbf{E}^2(\mathbb{T}\times\mathbb{R})$ associated with the Hamiltonian structure of (2.2.1) :

$$||u||_{\mathbf{E}^2}^2 = ||u||_{L^2}^2 + \left|\left|\partial_x^2 u\right|\right|_{L^2}^2 + \left|\left|\partial_x^{-1}\partial_y u\right|\right|_{L^2}^2 .$$

However, they discovered a resonant low-high frequency interaction which causes the failure of the bilinear estimate in Bourgain spaces when (2.2.1) is considered on $\mathbb{T}^2$. Just as the standard KP-I equation, this obstruction is not just technical since it leads to the

**Proposition 2.2.1**

*Let* $(s_1, s_2) \in \mathbb{R}^2$. *Then there exists a period* $\lambda > 0$ *such that for any* $T > 0$, *the flow map* $\Phi_{t,\lambda}$ *of (2.2.1) fails to be* $\mathcal{C}^2$*-differentiable at 0 from* $H^{s_1,s_2}(\mathbb{T}_\lambda^2)$ *to* $H^{s_1,s_2}(\mathbb{T}_\lambda^2)$, *where* $\mathbb{T}_\lambda^2 = \mathbb{T}\times\lambda^{-1}\mathbb{T}$. *Again, this result is also true in the energy space* $\mathbf{E}^2(\mathbb{T}_\lambda^2)$.

This implies that, at least for some torus $\mathbb{T}_\lambda^2$, the iteration methods will not work. This is in contrast with the standard KP-I equation, which is quasilinear whatever the domain is, whereas (2.2.1) has a radically different behaviour if the functions are periodic in $y$ or not. In particular there is a substantial gap of regularity between the well-posedness theory on $\mathbb{R}^2$ or on $\mathbb{T}_\lambda^2$.

### 2.2.2 The results on the quasilinear equation

By implementing the refined energy method (subsection 1.2.5), Ionescu and Kenig [IK07] have proved that (2.2.1) is globally well-posed in the energy space $\mathbf{E}^2(\mathbb{R}\times\mathbb{T})$. Their result relies on a small time Strichartz type estimate

$$\left|\left|\mathrm{e}^{t\mathcal{L}}P_M^x P_N^y u_0\right|\right|_{L^2(|t|\lesssim M^{-2})L^\infty_{xy}} \lesssim (1\vee NM^{-3})^{1/2}M^{(-1/2)+}\,||u_0||_{L^2} \tag{2.2.2}$$

which is proved by a $TT^*$ argument since the kernel is an oscillating integral in the dual variable of $x$ and an exponential sum in the transverse variable. With the Poisson formula one can get rid of the latter which is transformed into a Gaussian oscillating integral (modulo acceptable error) which is computed explicitly, and the former is estimated by a standard argument. Note that the factor $M^{(-1/2)+}$ coming from this former integral reflects the high dispersion effect in the $x$ direction. However, when the data is also periodic in $x$ we have to estimate an exponential sum, which spoils this dispersive effect so that we obtain the same estimate as (2.2.2) with a loss

$$\left|\left|\mathrm{e}^{t\mathcal{L}}P_M^x P_N^y u_0\right|\right|_{L^2(|t|\lesssim M^{-2})L^\infty_{xy}} \lesssim (1\vee NM^{-3})^{1/2}M^{(15/32)+}\,||u_0||_{L^2} \tag{2.2.3}$$

Plugging this estimate in the refined energy method restricts the well-posedness regularity to $s > 2 + 15/32$ which is worse than the standard energy method for quasilinear equations.

Thus, when the data is periodic in both variables, the only local well-posedness result were that of Iório and Nunes [IN98] in $H^s(\mathbb{T}^2)$ when $s > 2$. In order to get well-posedness in a larger space allowing to use the conservation laws to globalize the solutions, we then have to implement the short time Fourier restriction norm method. As for the standard KP-I equation, it relies on estimating the resonant contribution

$$||\partial_x \mathfrak{R}(u_{M_1} v_{M_2})||_{\mathbf{N}_M} \lesssim \Gamma_{\mathbb{T}^2}(M_1, M_2, M)\, ||u||_{\mathbf{F}_{M_1}} ||v||_{\mathbf{F}_{M_2}} \tag{2.2.4}$$

where in this case the small time intervals have size $M^{-2}$, and the loss is $\Gamma_{\mathbb{T}^2} = (M_1 \wedge M_2 \wedge M)^{-1/2}$. As far as $\mathbb{R}^2$ is concerned, Guo, Huo and Fang [GHF17] have proved that this estimate holds with $\Gamma_{\mathbb{R}^2}(M_1, M_2, M) = (M_1 \wedge M_2 \wedge M)^{-1/2}(M_1 \vee M_2 \vee M)^{-3/2}$. As we can see, the loss between $\Gamma_{\mathbb{T}^2}$ and $\Gamma_{\mathbb{R}^2}$ is very similar to that between (2.1.9) and (2.1.10) and actually is a consequence of the same phenomenon. Nevertheless, (2.2.4) is still sufficient to get the

**Theorem 2.2.2**

*The periodic fifth-order KP-I equation (2.2.1) is globally well-posed in the energy space* $\mathbf{E}^2(\mathbb{T}^2)$.

To conclude, we make a few remarks : first, this result is proved in chapter 4 on a square torus, yet the proof can easily be adapted for a general torus with any period, thus the method we employed is consistent with proposition 2.2.1. One can then ask whether it is ill-posed in the sense of definition 1.1.3. Let us highlight that a key point in the proof is the preliminar reduction to data with zero $x$ mean value (which is an invariant of the dynamics). Namely, we construct a flow map $\Phi_t^0$ defined on the subspace $\mathbf{E}_0^2(\mathbb{T}_\lambda^2)$ of data with zero $x$-mean value, and then we use the transformation

$$T_\theta^t := u_0(x, y) \mapsto u_0(x + \theta t, y) - \theta$$

to construct a flow on the whole energy space $\mathbf{E}^2(\mathbb{T}_\lambda^2)$ by the formula

$$\Phi_t := T_{-\theta(u_0)}^t \circ \Phi_t^0 \circ T_{\theta(u_0)}^0$$

taking $\theta(u_0) := \int_{\mathbb{T}} u_0(x, y)\mathrm{d}x$ (which is a quantity actually independent of $y$ and invariant under the equation). But for $t \in ]0; 1]$ the transformation $T_{\theta(\cdot)}^t$ is *not* uniformly continuous on any Sobolev space : indeed, take $n \in \mathbb{N}$ and $s_1, s_2 \in \mathbb{R}$ and define

$$u_n(x, y) := n^{-s_1} \cos(nx) + n^{-1} \text{ and } v_n(x, y) := n^{-s_1} \cos(nx)$$

Then $u_n, v_n \in H^{s_1, s_2}(\mathbb{T}^2)$ with

$$||u_n - v_n||_{H^{s_1,s_2}} = cn^{-1} \underset{n \to +\infty}{\longrightarrow} 0$$

but

$$\big|\big|T_{\theta(u_n)}(u_n) - T_{\theta(v_n)}(v_n)\big|\big|_{H^{s_1,s_2}} = \big|\big|n^{-s_1} \cos\big(n[x + n^{-1}t]\big) - n^{-s} \cos(nx)\big|\big|_{H^{s_1,s_2}} \gtrsim |\sin(t/2)| > 0$$

as soon as $t \in ]0; 1]$. This is a very general fact : the same proof works for any periodic Hamiltonian equation under the form (1.1.11), for example for the KdV equation. However on $L^2(\mathbb{T})$ this equation is known to be semilinearly well-posed on the hyperplane of data with zero mean value [Bou93a] (see also [KT06, Appendice A]). By the way, the proof of [KT08] and of theorem 2.1.4 relies on a localized version of this transform (in the case of non periodic variables). Thus a real expression of the quasilinear behaviour would be to prove the semilinear ill-posedness even for data with prescribed mean value.

The final remark is that it may be possible to adapt the proof of theorem 2.1.1 to show that the same resonant estimate as for (2.2.1) on $\mathbb{R}^2$ holds on $\mathbb{R}\times\mathbb{T}$ (that is $\Gamma_{\mathbb{R}\times\mathbb{T}}(M_1, M_2, M) = \Gamma_{\mathbb{R}^2}(M_1, M_2, M)$ with our previous notations) which would yield global well-posedness in $L^2(\mathbb{R} \times \mathbb{T})$ (see remark 4.3.7).

# Chapter 3

# The short-time Fourier restiction norm method for the KP-I equation on a cylinder

This chapter essentially contains the article [Rob18], to appear in *Annales de l'Institut Henri Poincaré (C) Analyse non-linéaire.*

## 3.1 Introduction

### 3.1.1 Motivations

The Kadomtsev-Petviashvili equations

$$\partial_t u + \partial_x^3 u + \epsilon \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0 \tag{3.1.1}$$

were first introduced in [KP70] as two-dimensional generalizations of the Korteweig-de Vries equation

$$\partial_t u + \partial_x^3 u + u \partial_x u = 0 \tag{3.1.2}$$

They model long, weakly nonlinear waves propagating essentially along the $x$ direction with a small dependence in the $y$ variable. The coefficient $\epsilon \in \{-1; 1\}$ takes into account the surface tension. When this latter is strong ($\epsilon = -1$), (3.1.1) is then called KP-I equation, whereas KP-II equation refers to a small surface tension ($\epsilon = +1$).

The KdV equation (3.1.2) admits a particular family of traveling waves solutions, the so-called solitons $Q_c(x - ct)$ with speed $c > 0$ :

$$Q_c(x) := 3c \cdot \cosh\left(\frac{\sqrt{c}}{2} x\right)^{-2}$$

From the work of Benjamin [Ben72], we know that these solutions are orbitally stable in $H^1(\mathbb{R})$ under the flow generated by the KdV equation (3.1.2), meaning that every solution of (3.1.2) with initial data close to $Q_c$ in $H^1(\mathbb{R})$ remains close in $H^1(\mathbb{R})$ to the $Q_c$-orbit (under the action of translations) at any time $t > 0$.

Looking at (3.1.1), we see that every solution of the KdV equation (3.1.2) is a solution of the KP equations (3.1.1), seen as a function independent of $y$. It is then a natural question to ask whether $Q_c$ is orbitally stable or unstable under the flow generated by (3.1.1). In order to do so, we first need a global well-posedness theory for (3.1.1) in a space containing $Q_c$. In particular, this rules out any well-posedness result in an anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$. A more suited space to look for is the energy space for functions periodic in $y$ :

$$\mathbf{E}(\mathbb{R}\times\mathbb{T}) := \left\{u_0(x,y)\in L^2(\mathbb{R}\times\mathbb{T}),\quad \partial_x u_0\in L^2(\mathbb{R}\times\mathbb{T}),\quad \partial_x^{-1}\partial_y u_0\in L^2(\mathbb{R}\times\mathbb{T})\right\} \tag{3.1.3}$$

where $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$. Indeed, due to the Hamiltonian structure of (3.1.1), the mass

$$\mathcal{M}(u)(t) := \int_{\mathbb{R}\times\mathbb{T}} u^2(t,x,y)\mathrm{d}x\mathrm{d}y \tag{3.1.4}$$

and the energy

$$\mathcal{E}(u)(t) := \int_{\mathbb{R}\times\mathbb{T}} \left\{(\partial_x u)^2(t,x,y) + (\partial_x^{-1}\partial_y u)^2(t,x,y) - \frac{1}{3}u^3(t,x,y)\right\}\mathrm{d}x\mathrm{d}y \tag{3.1.5}$$

are (at least formally) conserved by the flow, i.e. $\mathcal{M}(u)(t)=\mathcal{M}(u)(0)$ and $\mathcal{E}(u)(t)=\mathcal{E}(u)(0)$, for any time $t$ and any solution $u$ of the KP-I equation defined on $[0,t]$. The conservation of the energy allows one to extend local solutions in $\mathcal{C}([-T,T],\mathbf{E})$ into solutions globally defined. In this article, we thus focus on the following Cauchy problem for the KP-I equation set on $\mathbb{R}\times\mathbb{T}$ :

$$\begin{cases}\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, & (t,x,y)\in\mathbb{R}^2\times\mathbb{T}\\ u(t=0)=u_0\in\mathbf{E}(\mathbb{R}\times\mathbb{T})\end{cases} \tag{3.1.6}$$

### 3.1.2 Well-posedness results

The KP equations (3.1.1) have been extensively studied in the past few decades. Using a standard energy method, Iório and Nunes [IN98] proved existence and uniqueness of zero mean value solutions in $H^s$, $s>2$, for both KP equations on $\mathbb{R}^2$ and $\mathbb{T}^2$. From the point of view of well-posedness, the KP-II equation is much better understood. Indeed, since the pioneering work of Bourgain [Bou93b], we know that the KP-II equation is globally well-posed on both $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{T}^2)$. On $\mathbb{R}^2$, Takaoka and Tzvetkov [TT01] and Isaza and Mejia [IM01] pushed the low regularity local well-posedness theory down to the anistropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1>-1/3$, $s_2\geqslant 0$. Later, Hadac [Had08] and then Hadac, Herr and Koch [HHK09] reached the threshold $s_1\geqslant -1/2$, $s_2\geqslant 0$ which is the scaling critical regularity for the KP-II equation. As for the initial value problem on $\mathbb{R}\times\mathbb{T}$, in order to study the stability of the KdV soliton under the flow of the KP-II equation, Molinet, Saut and Tzvetkov [MST11] proved global well-posedness on $L^2(\mathbb{R}\times\mathbb{T})$.

The situation is radically different regarding the Cauchy theory for the KP-I equation. From the work of Molinet, Saut and Tzvetkov [MST02b], we know that this equation badly behaves with respect to pertubation methods. In particular, it is not possible to get well-posedness of (3.1.6) using the standard Fourier restriction norm method of Bourgain, nor any method using a fixed point argument on the Duhamel formula associated with (3.1.6) since Koch and Tzvetkov [KT08] proved that on $\mathbb{R}^2$, the flow map even fails to be uniformly continuous on bounded sets of $\mathcal{C}([-T,T],\mathbf{E})$. It is thus expected to have the same ill-posedness result on $\mathbb{R}\times\mathbb{T}$. Using the refined energy method introduced in [KT03], Kenig [Ken04], and then Ionescu and Kenig [IK07] proved global well-posedness in the "second energy space"

$$Z^2 =: \left\{u\in L^2,\ \partial_x^2 u\in L^2,\ \partial_x^{-2}\partial_y^2 u\in L^2\right\}$$

for functions on $\mathbb{R}^2$, and both $\mathbb{R}\times\mathbb{T}$ and $\mathbb{T}^2$, respectively. Lately, Ionescu, Kenig and Tataru [IKT08] introduced the so-called short time Fourier restriction norm method and were able to prove global well-posedness of the KP-I equation in the energy space $\mathbf{E}(\mathbb{R}^2)$. Zhang [Zha15] adapted this method in the periodic setting and got local well-posedness in the Besov space $\mathbf{B}^1_{2,1}(\mathbb{T}^2)$, which is almost the energy space but still strictly embedded in it. Overcoming the logarithmic divergence that appears in [Zha15] to reach the energy space $\mathbf{E}(\mathbb{T}^2)$ is still an important open problem. In our case, we prove the following theorem, which answers the global well-posedness issue in the partially periodic setting :

**Theorem 3.1.1**

*(a)* Global well-posedness for smooth data
*Take $u_0\in\mathbf{E}^\infty(\mathbb{R}\times\mathbb{T})$. Then, (3.1.6) admits a unique global solution*

$$u=\Phi^\infty(u_0)\in\mathcal{C}\left(\mathbb{R},\mathbf{E}^\infty(\mathbb{R}\times\mathbb{T})\right)$$

*which defines a flow map*

$$\Phi^\infty:\mathbf{E}^\infty(\mathbb{R}\times\mathbb{T})\to\mathcal{C}\left(\mathbb{R},\mathbf{E}^\infty(\mathbb{R}\times\mathbb{T})\right)$$

*In addition, for any $T>0$ and $\alpha\in\mathbb{N}^*$,*

$$||\Phi^\infty(u_0)||_{L^\infty_T\mathbf{E}^\alpha}\leqslant C(T,\alpha,||u_0||_{\mathbf{E}^\alpha}) \tag{3.1.7}$$

*(b)* Global well-posedness in the energy space
*For any $u_0\in\mathbf{E}(\mathbb{R}\times\mathbb{T})$ and $T>0$, there exists a unique solution $u$ to (3.1.6) in the class*

$$\mathcal{C}([-T;T],\mathbf{E})\cap\mathbf{F}(T)\cap\mathbf{B}(T) \tag{3.1.8}$$

*Moreover, the corresponding global flow*

$$\Phi^1:\mathbf{E}\to\mathcal{C}(\mathbb{R},\mathbf{E})$$

*is continuous and leaves $\mathcal{M}$ and $\mathcal{E}$ invariants.*

The function spaces $\mathbf{E}^\alpha$, $\mathbf{E}^\infty$, $\mathbf{F}$ and $\mathbf{B}$ are defined in section 3.4 below.

### 3.1.3 Stability results

As far as stability issues are concerned, Mizumachi and Tzvetkov [MT12] proved that the KdV line soliton is stable under the flow generated by the KP-II equation on $L^2(\mathbb{R}\times\mathbb{T})$ for any speed $c>0$. Regarding the KP-I equation, Rousset and Tzvetkov [RT12] proved that $Q_c$ is orbitally unstable in $\mathbf{E}^1(\mathbb{R}\times\mathbb{T})$ under the KP-I flow constructed on $Z^2(\mathbb{R}\times\mathbb{T})$ in [IK07], whenever $c>c^*=4/\sqrt{3}$, and that it is orbitally stable if $c<c^*$. Thus, as a byproduct of [RT12] and of our theorem 3.1.1, we can extend the range of admissible perturbations in [RT12, Theorem 1.4] to get

**Corollary 3.1.2**

*Assume $c<4/\sqrt{3}$, then $Q_c$ is orbitally stable in $\mathbf{E}$.*
*More precisely, for every $\varepsilon>0$, there exists $\delta>0$ such that for every $u_0\in\mathbf{E}(\mathbb{R}\times\mathbb{T})$*

*satisfying*

$$||u_0 - Q_c||_{\mathbf{E}(\mathbb{R}\times\mathbb{T})} < \delta$$

*we have*

$$\sup_{t\in\mathbb{R}} \inf_{a\in\mathbb{R}} \left|\left|\Phi^1(u_0)(t, x-a, y) - Q_c(x-ct)\right|\right|_{\mathbf{E}(\mathbb{R}\times\mathbb{T})} < \varepsilon$$

The proof of corollary 3.1.2 is a straightforward adaptation of the argument in [RT12]. Indeed, the proof of [RT12, Theorem 1.4] only uses the extra conditions $\partial_x^2 u \in L^2$, $\partial_x^{-2}\partial_y^2 u \in L^2$ to have the global solutions from [IK07]. For the sake of completeness, we present the outlines of the proof in section 3.11.

### 3.1.4 Strategy of the proof

Let us now briefly discuss the main ingredients in the proof of theorem 3.1.1.

As pointed out above, it is irrelevant to look for functions spaces $\mathbf{F}(T) \hookrightarrow \mathcal{C}([-T,T],\mathbf{E})$ and $\mathbf{N}(T)$ such that any solution to (3.1.6) satisfies

1. a linear estimate
$$||u||_{\mathbf{F}(T)} \lesssim ||u_0||_{\mathbf{E}} + \left|\left|\partial_x(u^2)\right|\right|_{\mathbf{N}(T)} \tag{3.1.9}$$
2. a bilinear estimate
$$||\partial_x(uv)||_{\mathbf{N}(T)} \lesssim ||u||_{\mathbf{F}(T)} \, ||v||_{\mathbf{F}(T)} \tag{3.1.10}$$

In order to construct solutions in $\mathbf{E}$, we will thus use the functions spaces $\mathbf{F}(T)$, $\mathbf{N}(T)$ and $\mathbf{B}(T)$ introduced in [IKT08]. Those spaces are built to combine the idea introduced in [KT03] of a priori estimates on short times (depending on the frequency) for frequency localized solutions, with the standard Bourgain spaces $X^{s,b}$ of [Bou93b]. Thus, we will replace (3.1.9)-(3.1.10) with

1. a linear estimate
$$||u||_{\mathbf{F}(T)} \lesssim ||u||_{\mathbf{B}(T)} + \left|\left|\partial_x(u^2)\right|\right|_{\mathbf{N}(T)} \tag{3.1.11}$$
2. a bilinear estimate
$$||\partial_x(uv)||_{\mathbf{N}(T)} \lesssim ||u||_{\mathbf{F}(T)} \, ||v||_{\mathbf{F}(T)} \tag{3.1.12}$$
3. an energy estimate
$$||u||^2_{\mathbf{B}(T)} \lesssim ||u_0||^2_{\mathbf{E}} + ||u||^3_{\mathbf{F}(T)} \tag{3.1.13}$$

With (3.1.11)-(3.1.12)-(3.1.13) at hand, we will get the existence part of theorem 3.1.1 from a standard continuity argument.

To get uniqueness, we will prove the analogue of (3.1.11)-(3.1.12)-(3.1.13) for the difference equation, at the $L^2$ level :

$$||u-v||_{\overline{\mathbf{F}}(T)} \lesssim ||u-v||_{\overline{\mathbf{B}}(T)} + ||\partial_x\{(u-v)(u+v)\}||_{\overline{\mathbf{N}}(T)} \tag{3.1.14}$$
$$||\partial_x\{(u-v)(u+v)\}||_{\overline{\mathbf{N}}(T)} \lesssim ||u-v||_{\overline{\mathbf{F}}(T)} \, ||u+v||_{\mathbf{F}(T)} \tag{3.1.15}$$
$$||u-v||^2_{\overline{\mathbf{B}}(T)} \lesssim ||u_0-v_0||^2_{L^2} + ||u+v||_{\mathbf{F}(T)} \, ||u-v||^2_{\overline{\mathbf{F}}(T)} \tag{3.1.16}$$

The main technical difficulties, compared to the case of $\mathbb{R}^2$, are the lack of a scale-invariant Strichartz estimate, and the impossibility to make the change of variables as in the proof of [IKT08, Lemma 5.1 (a)] to estimate the volume of the resonant set. The first one is handled with frequency localized Strichartz estimates in the spirit of [Bou93b, MST11]. For the second one, we follow Zhang [Zha15, Lemma 3.1], but looking closely on the computations we are able to take advantage of the smallness of the intervals in which the frequency for the $x$ variable varies (note that this is not possible in [Zha15] since this frequency lives in $\mathbb{Z}$) and to recover the same estimate as in [IKT08] in this case. We also use a weighted Bourgain type space to deal with the logarithmic divergence in the energy estimate.

### 3.1.5 Organization of the chapter

Sections 3.2 and 3.4 introduce general notations as well as functions spaces. In section 3, we first prove proposition 2.1.3. We begin the proof of theorem 3.1.1 in section 3.5 by proving estimate (3.1.11). After establishing some general dyadic estimates in section 3.6, sections 3.7 and 3.8 deal with (3.1.12) and (3.1.13) respectively. The proof of theorem 3.1.1 is then completed in section 3.9. Finally, in the last sections 3.10 and 3.11 we prove proposition 2.1.4 and we we recall the arguments to obtain corollary 3.1.2.

## 3.2 Notations

— We use the notations of [Mol07] to deal with Fourier transform of periodic functions with a large period $2\pi\lambda > 0$. Let $\lambda \geqslant 1$ be fixed. We define $(\mathrm{d}q)_\lambda$ to be the renormalized counting measure on $\lambda^{-1}\mathbb{Z}$ :

$$\int u(q)(\mathrm{d}q)_\lambda := \frac{1}{\lambda}\sum_{q\in\lambda^{-1}\mathbb{Z}} u(q)$$

In the sequel, all the Lebesgue norms in $q$ will be with respect to $(\mathrm{d}q)_\lambda$. Moreover, the space-time Lebesgue norms are defined as

$$||f||_{L^p_{\xi,q}L^r_\tau} := \left\{\int_{\mathbb{R}\times\lambda^{-1}\mathbb{Z}}\left(\int_{\mathbb{R}}|f|^r\mathrm{d}\tau\right)^{p/r}\mathrm{d}\xi(\mathrm{d}q)_\lambda\right\}^{1/p}$$

For a $2\pi\lambda$-periodic function $f$, we define its Fourier transform as

$$\widehat{f}(q) := \int_0^{2\pi\lambda}\mathrm{e}^{-iqx}f(y)\mathrm{d}y,\ q\in\lambda^{-1}\mathbb{Z}$$

and we have the inversion formula

$$f(y) = \int \mathrm{e}^{iqy}\widehat{f}(q)(\mathrm{d}q)_\lambda$$

We write $\mathbb{T}_\lambda := \mathbb{R}/2\pi\lambda\mathbb{Z}$. Whenever $\lambda = 1$ we drop the lambda.

— The Fourier transform of a function $u_0(x,y)$ on $\mathbb{R}\times\mathbb{T}_\lambda$ or $u(t,x,y)$ on $\mathbb{R}^2\times\mathbb{T}_\lambda$ is denoted $\widehat{u}$ or $\mathcal{F}u$ :

$$\widehat{u_0}(\xi,q) := \int_{\mathbb{R}\times\mathbb{T}_\lambda}\mathrm{e}^{-i(\xi x+qy)}u_0(x,y)\mathrm{d}x\mathrm{d}y,\ (\xi,q)\in\mathbb{R}\times\lambda^{-1}\mathbb{Z}$$

and

$$\widehat{u}(\tau,\xi,q) := \int_{\mathbb{R}^2\times\mathbb{T}_\lambda}\mathrm{e}^{-i(\tau t+\xi x+qy)}u(t,x,y)\mathrm{d}t\mathrm{d}x\mathrm{d}y,\ (\tau,\xi,q)\in\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}$$

$\mathcal{F}_t u$ stands for the partial Fourier transform of $u(t,x,y)$ with respect to $t$, whereas $\mathcal{F}_{xy}u$ means the partial Fourier transform of $u$ with respect to space variables $x, y$, and similarly for $\mathcal{F}_x$, $\mathcal{F}_y$.

We always note $(\tau,\xi,q)\in\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}$ the Fourier variables associated with $(t,x,y)\in\mathbb{R}^2\times\mathbb{T}_\lambda$. We note eventually $\zeta = (\xi,q)\in\mathbb{R}\times\lambda^{-1}\mathbb{Z}$.

— We denote $\star$ the convolution product for functions on $\mathbb{R}$ or $\lambda^{-1}\mathbb{Z}$ : to specify the variables,

$$f(x') \star_x g(x') \text{ means } (f \star g)(x) = \int_{\mathbb{R} \text{ or } \lambda^{-1}\mathbb{Z}} f(x-x')g(x')\mathrm{d}x'$$

— We use the "bracket" notation $\langle\cdot\rangle$ for the weight in the definition of inhomogeneous Sobolev spaces, i.e

$$\langle\xi\rangle^s := \left(1+\xi^2\right)^{s/2}$$

— $U(t)$ is the unitary group defined by the linear evolution equation associated with (3.1.6) :

$$\forall u_0 \in L^2(\mathbb{R}\times\mathbb{T}_\lambda), \quad \widehat{U(t)u_0}(\xi,q) = \mathrm{e}^{it\omega(\xi,q)}\widehat{u_0}(\xi,q)$$

where

$$\omega(\xi,q) := \xi^3 + q^2/\xi$$

We also note

$$\sigma(\tau,\xi,q) := \tau - \omega(\xi,q) = \tau - \xi^3 - \frac{q^2}{\xi}$$

the modulation associated with (3.1.6).

— For positive reals $a$ and $b$, $a \lesssim b$ means that there exists a positive constant $c > 0$ (independent of the various parameters, including $\lambda$) such that $a \leqslant c \cdot b$.
The notation $a \sim b$ stands for $a \lesssim b$ and $b \lesssim a$.

— We note $M \in \mathbb{R}_+^*$ (respectively $K \geqslant 1$) the dyadic frequency decomposition of $|\xi|$ (respectively of $\langle\sigma\rangle$), i.e $M \in 2^{\mathbb{Z}}$ and $K \in 2^{\mathbb{N}}$.
We define then

$$D_{\lambda,M,K} := \left\{(\tau,\xi,q) \in \mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}, \quad |\xi| \sim M, \langle\sigma(\tau,\xi,q)\rangle \sim K\right\}$$

and

$$D_{\lambda,M,\leqslant K} := \left\{(\tau,\xi,q) \in \mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}, \quad |\xi| \sim M, \langle\sigma(\tau,\xi,q)\rangle \lesssim K\right\} = \bigcup_{K' \leqslant K} D_{\lambda,M,K'}$$

We note also

$$\mathfrak{I}_M := \{M/2 \leqslant |\xi| \leqslant 3M/2\}$$

and

$$\mathfrak{I}_{\leqslant M} := \{|\xi| \leqslant 3M/2\} = \bigcup_{M' \leqslant M} \mathfrak{I}_{M'}$$

— We use the notations $M_1 \wedge M_2 := \min(M_1,M_2)$ and $M_1 \vee M_2 := \max(M_1,M_2)$.
For $M_1, M_2, M_3 \in \mathbb{R}_+^*$, $M_{\min} \leqslant M_{med} \leqslant M_{max}$ denotes the increasing rearrangement of $M_1, M_2, M_3$, i.e

$$M_{min} := M_1 \wedge M_2 \wedge M_3, \quad M_{max} = M_1 \vee M_2 \vee M_3$$
$$\text{and } M_{med} = M_1 + M_2 + M_3 - M_{max} - M_{min}$$

— We use two different Littlewood-Paley decompositions : the first one is homogeneous (on $2^{\mathbb{Z}}$) for $|\xi|$, the last one is inhomogeneous for $\langle\sigma\rangle \in 2^{\mathbb{N}}$.
Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $0 \leqslant \chi \leqslant 1$, $\mathrm{supp}\chi \subset [-8/5; 8/5]$ and $\chi \equiv 1$ on $[-5/4; 5/4]$.

    — For $M \in 2^{\mathbb{Z}}$, we then define $\eta_M(\xi) := \chi(\xi/M) - \chi(2\xi/M)$, such that $\text{supp}\eta_M \subset \{5/8M \leqslant |\xi| \leqslant 8/5M\}$ and $\eta_M \equiv 1$ on $\{4/5M \leqslant |\xi| \leqslant 5/4M\}$. Thus $\xi \in \text{supp}\eta_M \Rightarrow \xi \in \mathfrak{I}_M$ and $|\xi| \sim M$.
    — For $K \in 2^{\mathbb{N}}$, we also define $\rho_1(\sigma) := \chi(\sigma)$ and $\rho_K(\sigma) := \chi(\sigma/K) - \chi(2\sigma/K)$, $K > 1$, such that $\text{supp}\rho_K \subset \{5/8K \leqslant |\sigma| \leqslant 8/5K\}$ and $\rho_K \equiv 1$ on $\{4/5M \leqslant |\sigma| \leqslant 5/4K\}$, $K > 1$. Thus $\sigma \in \text{supp}\rho_K \Rightarrow \langle\sigma\rangle \sim K$.
    — When needed, we may use other decompositions $\widetilde{\chi}$, $\widetilde{\eta}$ and $\widetilde{\rho}$ with the similar properties as $\chi$, $\eta$, $\rho$ and satisfying $\widetilde{\chi} \equiv 1$ on $\text{supp}\chi$, $\widetilde{\eta} \equiv 1$ on $\text{supp}\eta$ and $\widetilde{\rho} \equiv 1$ on $\text{supp}\rho$.
    — Finally, for $\kappa \in \mathbb{R}_+^*$, we note $\chi_\kappa(x) := \chi(x/\kappa)$.
— We also define the Littlewood-Paley projectors associated with the previous decompositions :

$$P_M u := \mathcal{F}^{-1}\left(\eta_M(\xi)\widehat{u}\right) \text{ and } P_{\leqslant M} u := \sum_{M' \leqslant M} P_M u = \mathcal{F}^{-1}\left(\chi_M(\xi)\widehat{u}\right)$$

Moreover, we define

$$P_{Low} := P_{\leqslant 2^{-5}} \text{ and } P_{High} := 1 - P_{Low}$$

— The energy space $\mathbf{E}_\lambda$ is defined as in (3.1.3) for any period $2\pi\lambda$ :

$$\mathbf{E}(\mathbb{R} \times \mathbb{T}_\lambda) := \left\{u_0 \in L^2(\mathbb{R} \times \mathbb{T}_\lambda),\ \ \partial_x u_0 \in L^2(\mathbb{R} \times \mathbb{T}_\lambda),\ \partial_x^{-1}\partial_y u_0 \in L^2(\mathbb{R} \times \mathbb{T}_\lambda)\right\}$$

It is endowed with the norm

$$||u_0||_{\mathbf{E}_\lambda} := ||\langle\xi\rangle \cdot p(\xi, q) \cdot \widehat{u_0}||_{L^2}$$

i.e $\mathbf{E}_\lambda$ is a weighted Sobolev space, with the weight defined as

$$p(\xi, q) := \left\langle \langle\xi\rangle^{-1} q/\xi \right\rangle,\ (\xi, q) \in \mathbb{R} \times \lambda^{-1}\mathbb{Z} \tag{3.2.1}$$

so that

$$|\langle\xi\rangle \cdot p(\xi, q)|^2 = 1 + \xi^2 + \frac{q^2}{\xi^2} \tag{3.2.2}$$

i.e

$$||u_0||_{\mathbf{E}_\lambda}^2 = ||u_0||_{L^2}^2 + ||\partial_x u_0||_{L^2}^2 + \left|\left|\partial_x^{-1}\partial_y u_0\right|\right|_{L^2}^2$$

More generally, for $\alpha \in \mathbb{N}$, we define

$$\mathbf{E}_\lambda^\alpha := \left\{u_0(x, y) \in L^2(\mathbb{R} \times \mathbb{T}_\lambda),\ \ ||u_0||_{\mathbf{E}_\lambda^\alpha} := ||\langle\xi\rangle^\alpha \cdot p(\xi, q) \cdot \widehat{u_0}||_{L^2} < +\infty\right\} \tag{3.2.3}$$

and

$$\mathbf{E}_\lambda^\infty = \bigcap_{\alpha \in \mathbb{N}^*} \mathbf{E}_\lambda^\alpha \tag{3.2.4}$$

— For a real $\xi$, we define

$$[\xi]_\lambda := \lambda^{-1}\lfloor\lambda\xi\rfloor \in \lambda^{-1}\mathbb{Z}$$

— For a set $A \subset \mathbb{R}^d$, $\mathbb{1}_A$ is the characteristic function of $A$ and if $A$ is Lebesgue-measurable, $|A|$ means its measure. Similarly, if $A \subset \lambda^{-1}\mathbb{Z}$, its measure with respect to $(\mathrm{d}q)_\lambda$ will also be noted $|A|$. When $A \subset \mathbb{Z}$ is a finite set, its cardinal is denoted $\#A$.
— For $M > 0$ and $s \in \mathbb{R}$, $\lesssim M^{s-}$ means $\leqslant C_\varepsilon M^{s-\varepsilon}$ for any choice of $\varepsilon > 0$ small enough. We define similarly $M^{s+}$.

## 3.3 Failure of the bilinear estimate in the standard Bourgain space

In this section we prove proposition 2.1.3, thus showing that we cannot use the contraction principle to solve (3.1.6). Indeed, since the nonlinearity depends analytically upon $u$, using a Picard iteration scheme on the Duhamel formula would yield an analytic flow map data-solution (see [Bou93a]).

The proof of this result is a straightforward adaptation of [MST02b, Theorem 5.1] to the case of partially periodic data : it exploits the same resonant interaction between low and high frequencies in the nonlinear term.

Indeed, proceeding by contradiction, let us assume that there exists $T > 0$ and $t \in [-T;T]$ being such that the flow map $\Phi_t$ is $\mathcal{C}^2$. Then the map $\gamma \in \mathbb{R} \mapsto \Phi_t(\gamma\varphi)$ is $\mathcal{C}^2$ as well and admits the Taylor expansion

$$\Phi_t(\gamma\varphi) = \gamma\,(\partial_\gamma)_{|\gamma=0}\,\Phi_t(\gamma\varphi) + \frac{\gamma^2}{2}\,(\partial_\gamma)^2_{|\gamma=0}\,\Phi_t(\gamma\varphi) + o(\gamma^2)$$

Moreover, as $\Phi_t$ is $\mathcal{C}^2$, we have a bound

$$\left|\left|(\partial_\gamma)^2_{|\gamma=0}\,\Phi_t(\gamma\varphi)\right|\right|_{H^{s_1,s_2}} \lesssim ||\varphi||_{H^{s_1,s_2}} \tag{3.3.1}$$

Now, recall that $\Phi_t(\gamma\varphi)$ is the unique solution at time $t$ to (3.1.6) with initial data $\gamma\varphi$, thus it satisfies the Duhamel formula

$$\Phi_t(\gamma\varphi) = \gamma U(t)\varphi + \int_0^t U(t-t')\Phi_{t'}(\gamma\varphi)\partial_x\Phi_{t'}(\gamma\varphi)\mathrm{d}t'$$

Now a direct computation gives

$$(\partial_\gamma)_{|\gamma=0}\,\Phi_t(\gamma\varphi) = U(t)\varphi =: u_1(t,x,y)$$

and

$$(\partial_\gamma)^2_{|\gamma=0}\,\Phi_t(\gamma\varphi) = -\int_0^t U(t-t')\partial_x(u_1^2)\mathrm{d}t' =: u_2(t,x,y)$$

As explained above, the contradiction to (3.3.1) will be raised by choosing the initial data as a resonant sum of a low and a high frequency piece.

Let us then set

$$\varphi(x,y) := \mathcal{F}^{-1}\left\{L+H\right\}$$

where

$$L(\xi,q) = \alpha^{-1/2}\mathbb{1}(\xi \in [\alpha/2;\alpha], q=0)$$

and

$$H(\xi,q) = \alpha^{-1/2}N^{-s_1-2s_2}\mathbb{1}(\xi \in [3^{-1/4}N; 3^{-1/4}N+\alpha], q=N^2)$$

are respectively the low and high frequency piece, meaning that we choose the parameters $0 < \alpha \ll 1 \ll N$, $N \in \mathbb{N}$. The amplitude of each piece is such that

$$||\varphi||_{H^{s_1,s_2}} \sim 1$$

Actually, one can take $\Re\varphi$ instead of $\varphi$ to work with real functions and check that the same argument as below applies (it just produces harmless lower order terms), so for the sake of simplicity we do not present this refinement.

Let us now compute $u_2$. As in the proof of [MST02b, Lemma 4], we use the formula

$$\int_0^t \mathrm{e}^{-it'\omega} f(t')\mathrm{d}t' = c\int_{\mathbb{R}} \frac{\mathrm{e}^{it(\tau-\omega)}-1}{\tau-\omega}\widehat{f}(\tau)\mathrm{d}\tau \tag{3.3.2}$$

which is true for any test function $f$ for which both sides of (3.3.2) are well-defined.

To prove (3.3.2), let us notice that both sides vanish at $t=0$ and that their derivative with respect to $t$ agree at any time.

Thus, using first the Fourier inversion formula in $x, y$ and then applying (3.3.2) leads to

$$u_2(t,x,y) = c\int_{\mathbb{R}^2}\sum_{q\in\mathbb{Z}} \xi e^{i(x\xi+qy+t\omega)}\frac{\mathrm{e}^{it(\tau-\omega)}-1}{\tau-\omega}\left(\widehat{u_1}\star\widehat{u_1}\right)\mathrm{d}\xi\mathrm{d}\tau$$

It remains to compute $\widehat{u_1}\star\widehat{u_1}$ :

$$\begin{aligned}(\widehat{u_1}\star\widehat{u_1})(\tau,\xi,q) &= \mathcal{F}\left\{U(t)\varphi\right\}\star\mathcal{F}\left\{U(t)\varphi\right\}\\ &= \left(\delta_0(\tau-\omega)\widehat{\varphi}\right)\star\left(\delta_0(\tau-\omega)\widehat{\varphi}\right)\\ &= \int_{\mathbb{R}}\sum_{q_1\in\mathbb{Z}}\delta_0(\tau-\omega(\xi_1,q_1)-\omega(\xi-\xi_1,q-q_1))\widehat{\varphi}(\xi_1,q_1)\widehat{\varphi}(\xi-\xi_1,q-q_1)\mathrm{d}\xi_1\end{aligned}$$

Now, in view of the definition of $\varphi$, we can write $u_2$ as the sum of 3 interactions : $L\times L$, $H\times H$ and $L\times H$. If we use the resonant function (see 3.6.9 below)

$$\Omega(\xi,q,\xi_1,q_1) := \omega(\xi_1,q_1)+\omega(\xi-\xi_1,q-q_1)-\omega(\xi,q)$$

and the bilinear functional

$$\begin{aligned}\mathcal{B}[f,g](t,\xi,q,\xi_1,q_1) := c\xi e^{it\omega(\xi,q)}\mathrm{e}^{it\Omega(\xi,q,\xi_1,q_1)/2}\cdot t\cdot\operatorname{sinc}\left(t\Omega(\xi,q,\xi_1,q_1)/2\right)&\\ f(\xi_1,q_1)g(\xi-\xi_1,q-q_1),&\end{aligned}$$

where sinc stands for the cardinal sine function

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}.$$

Then $\mathcal{F}_{x,y}u_2$ reads

$$\begin{aligned}\mathcal{F}_{x,y}u_2(t,\xi,q) = c\int_{\mathbb{R}}\sum_{q_1\in\mathbb{Z}}&\left\{\mathcal{B}[L,L](t,\xi,q,\xi_1,q_1)+\mathcal{B}[H,H](t,\xi,q,\xi_1,q_1)\right.\\ &\left.+2\mathcal{B}[L,H](t,\xi,q,\xi_1,q_1)\right\}\mathrm{d}\xi_1\\ &= f_1(t,\xi,q)+f_2(t,\xi,q)+f_3(t,\xi,q)\end{aligned}$$

We remark that for fixed $t>0$, $f_1, f_2, f_3$ are disjointly supported thanks to the definition of $L$ and $H$, thus

$$||u_2(t,.)||_{H^{s_1,s_2}} \geqslant \left|\left|\mathcal{F}_{xy}^{-1}f_3(t,.)\right|\right|_{H^{s_1,s_2}}$$

We are left with estimating $\Omega(\xi,q,\xi_1,q_1)$ on the support of $\mathcal{B}[L,H]$. In view of its definition, we can express the reonant function as

$$\Omega(\xi,q,\xi_1,q_1) = -\frac{\xi_1(\xi-\xi_1)}{\xi}\left\{\sqrt{3}\xi-\left(\frac{q_1}{\xi_1}-\frac{q-q_1}{\xi-\xi_1}\right)\right\}\left\{\sqrt{3}\xi+\left(\frac{q_1}{\xi_1}-\frac{q-q_1}{\xi-\xi_1}\right)\right\} \tag{3.3.3}$$

We get that for $(\xi, q, \xi_1, q_1) \in \text{supp}\mathcal{B}[L, H](t, .)$, the following etimate holds :

$$|\Omega(\xi, q, \xi_1, q_1)| \lesssim \alpha^2 N \tag{3.3.4}$$

Indeed, $q_1 = 0$, $q = q - q_1 = N^2$, $\xi_1 \in [\alpha/2; \alpha]$ and $\xi = \xi - \xi_1 + \xi_1 \in [3^{-1/4}N + \alpha/2; 3^{-1/4}N + 2\alpha]$, which leads to

$$\sqrt{3}\xi - \left(\frac{q_1}{\xi_1} - \frac{q - q_1}{\xi - \xi_1}\right) \leqslant 2\sqrt[4]{3}N \text{ and } \sqrt{3}\xi + \left(\frac{q_1}{\xi_1} - \frac{q - q_1}{\xi - \xi_1}\right) \leqslant \sqrt{3}^3\alpha + 2\sqrt[4]{3}^3\alpha^2 N^{-1}$$

Combining the estimates above with (3.3.3), we finally infer (3.3.4).

If we choose now $\alpha = N^{-(1+\varepsilon)/2}$, $0 < \varepsilon \ll 1$, we get at last the lower bound

$$|f_3(t, \xi, q)| \gtrsim |t| N^{1-s_1-2s_2} \mathbb{1}\left(\xi \in [3^{-1/4}N + \alpha/2; 3^{-1/4}N + 2\alpha], q = N^2\right)$$

which implies

$$||u_2(t)||_{H^{s_1,s_2}} \gtrsim |t|\alpha^{1/2}N = |t|N^{3/4-\varepsilon/4}$$

This raises a contradiction with (3.3.1) when taking $0 < \varepsilon \ll 1 \ll N$.

In particular, taking $s_1 = 1$ and $s_2 = 0$, we have $\varphi \in \mathbf{E}(\mathbb{R} \times \mathbb{T})$ with

$$||\varphi||_{\mathbf{E}} \sim ||\varphi||_{H^{1,0}} + \left|\left|\frac{q}{\xi}(L + H)\right|\right|_{L^2(\mathbb{R}\times\mathbb{Z})} \sim 1$$

and

$$||u_2(t)||_{\mathbf{E}} \geqslant \left|\left|\mathcal{F}_{xy}^{-1} f_3(t)\right|\right|_{\mathbf{E}} \geqslant \left|\left|\mathcal{F}_{xy}^{-1} f_3(t)\right|\right|_{H^{1,0}} \gtrsim |t|N^{3/4-} \to +\infty$$

so the same conclusion holds on $\mathbf{E}(\mathbb{R} \times \mathbb{T})$.

Thus, equation (3.1.6) cannot be treated with the standard Bourgain method, nor any contraction principle argument. In what follows, we use instead the short-time Fourier restriction norm method developped by Ionescu, Kenig and Tataru [IKT08], which is a compactness argument, to construct a global flow to (3.1.6).

# 3.4 Functions spaces

## 3.4.1 Definitions

Let $M \in 2^{\mathbb{Z}}$.

First, the dyadic energy space is defined as

$$E_{\lambda,M} := \left\{u_0 \in \mathbf{E}^0_\lambda,\ P_M u_0 = u_0\right\}$$

As in [IKT08], for $M \in 2^{\mathbb{Z}}$ and $b_1 \in [0; 1/2[$, the dyadic Bourgain type space is defined as

$$X^{b_1}_{\lambda,M} := \Big\{f(\tau, \xi, q) \in L^2(\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}), \text{supp} f \subset \mathbb{R} \times \mathfrak{I}_M \times \lambda^{-1}\mathbb{Z},$$
$$||f||_{X^{b_1}_{\lambda,M}} := \sum_{K \geqslant 1} K^{1/2}\beta^{b_1}_{M,K}\, ||\rho_K(\tau - \omega)f||_{L^2} < +\infty\Big\}$$

where the extra weight $\beta_{M,K}$ is

$$\beta_{M,K} := 1 \vee \frac{K}{(1 \vee M)^3}$$

This weight, already used in [Bou93b, MST11, GPWW11], allows to recover a bit of derivatives in the high modulation regime, thus preventing a logarithmic divergence in the energy estimate. Then, we use the $X^{b_1}_{\lambda,M}$ structure uniformly on time intervals of size $(1 \vee M)^{-1}$ :

$$F^{b_1}_{\lambda,M} := \{u(t,x,y) \in \mathcal{C}\left(\mathbb{R}, E_{\lambda,M}\right),\ P_M u = u, \\ ||u||_{F^{b_1}_{\lambda,M}} := \sup_{t_M \in \mathbb{R}} \left|\left|p \cdot \mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)u\right\}\right|\right|_{X^{b_1}_{\lambda,M}} < +\infty \Big\}$$

and

$$N^{b_1}_{\lambda,M} := \{u(t,x,y) \in \mathcal{C}\left(\mathbb{R}, E_{\lambda,M}\right),\ P_M u = u, \\ ||u||_{N^{b_1}_{\lambda,M}} := \sup_{t_M \in \mathbb{R}} \left|\left|(\tau - \omega + i(1\vee M))^{-1} p \cdot \mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)u\right\}\right|\right|_{X^{b_1}_{\lambda,M}} < +\infty \Big\}$$

For a function space $Y \hookrightarrow \mathcal{C}(\mathbb{R}, \mathbf{E}^\alpha_\lambda)$, we set

$$Y(T) := \left\{u \in \mathcal{C}\left([-T,T], \mathbf{E}^\alpha_\lambda\right),\ ||u||_{Y(T)} < +\infty\right\}$$

endowed with

$$||u||_{Y(T)} := \inf\left\{||\widetilde{u}||_Y,\quad \widetilde{u} \in Y,\quad \widetilde{u} \equiv u \text{ on } [-T,T]\right\} \tag{3.4.1}$$

Finally, the main function spaces are defined as

$$\mathbf{F}^{\alpha,b_1}_\lambda(T) := \{u \in \mathcal{C}([-T,T], \mathbf{E}^\alpha_\lambda), \\ ||u||_{\mathbf{F}^{\alpha,b_1}_\lambda(T)} := \left(\sum_{M>0} (1\vee M)^{2\alpha}\, ||P_M u||^2_{F^{b_1}_{\lambda,M}(T)}\right)^{1/2} < +\infty \Big\} \tag{3.4.2}$$

and

$$\mathbf{N}^{\alpha,b_1}_\lambda(T) := \{u \in \mathcal{C}([-T,T], \mathbf{E}^\alpha_\lambda), \\ ||u||_{\mathbf{N}^{\alpha,b_1}_\lambda(T)} := \left(\sum_{M>0} (1\vee M)^{2\alpha}\, ||P_M u||^2_{N^{b_1}_{\lambda,M}(T)}\right)^{1/2} < +\infty \Big\} \tag{3.4.3}$$

The last space is the energy space

$$\mathbf{B}^\alpha_\lambda(T) := \{u \in \mathcal{C}([-T,T], \mathbf{E}^\alpha_\lambda), \\ ||u||_{\mathbf{B}^\alpha_\lambda(T)} := \left(||P_{\leqslant 1} u_0||^2_{\mathbf{E}^\alpha_\lambda} + \sum_{M>1} \sup_{t_M \in [-T,T]} ||P_M u(t_M)||^2_{\mathbf{E}^\alpha_\lambda}\right)^{1/2} < +\infty \Big\} \tag{3.4.4}$$

For $b_1 = 1/8$, we drop the exponent.
If moreover $\alpha = 1$, we simply write $\mathbf{F}_\lambda(T)$, $\mathbf{N}_\lambda(T)$ et $\mathbf{B}_\lambda(T)$.

We define similarly the spaces

$$\overline{E_{\lambda,M}},\ \overline{F^{b_1}_{\lambda,M}},\ \overline{N^{b_1}_{\lambda,M}}$$

which are the equivalents of $E_{\lambda,M}$, $F^{b_1}_{\lambda,M}$, $N^{b_1}_{\lambda,M}$ but on an $L^2$ level, i.e without the weight $p(\xi,q)$. In particular,

$$||u||^2_{\mathbf{F}_\lambda(T)} \sim \sum_{M>0} (1\vee M)^2 \, ||u||^2_{\overline{F^{b_1}_{\lambda,M}}(T)} + \left|\left|\partial_x^{-1}\partial_y u\right|\right|^2_{\overline{F^{b_1}_{\lambda,M}}(T)} \tag{3.4.5}$$

For the difference equation, we will then use the $L^2$-type energy space

$$\left.\begin{aligned}\overline{\mathbf{B}_\lambda}(T) := \big\{u \in \mathcal{C}([-T;T], L^2(\mathbb{R}\times\mathbb{T}_\lambda)),\\ ||u||^2_{\overline{\mathbf{B}_\lambda}(T)} := ||P_{\leqslant 1}u_0||_{L^2} + \sum_{M>1}\sup_{t_M\in[-T;T]} ||P_M u(t_M)||^2_{L^2} < +\infty\end{aligned}\right\} \tag{3.4.6}$$

and the spaces for the difference of solutions and for the nonlinearity are

$$\left.\begin{aligned}\overline{\mathbf{F}_\lambda}^{b_1}(T) := \big\{u \in \mathcal{C}([-T;T], L^2(\mathbb{R}\times\mathbb{T}_\lambda)),\\ ||u||^2_{\overline{\mathbf{F}_\lambda}^{b_1}(T)} := \sum_{M>0} ||P_M u||^2_{\overline{F^{b_1}_{\lambda,M}}(T)} < +\infty\end{aligned}\right\} \tag{3.4.7}$$

and

$$\left.\begin{aligned}\overline{\mathbf{N}_\lambda}^{b_1}(T) := \big\{u \in \mathcal{C}([-T;T], L^2(\mathbb{R}\times\mathbb{T}_\lambda)),\\ ||u||^2_{\overline{\mathbf{N}_\lambda}^{b_1}(T)} := \sum_{M>0} ||P_M u||^2_{\overline{N^{b_1}_{\lambda,M}}(T)} < +\infty\end{aligned}\right\} \tag{3.4.8}$$

### 3.4.2 Basic properties

The following property of dyadic Bourgain type space is fundamental :

**Proposition 3.4.1**

*Let $M \in 2^{\mathbb{Z}}$, $b_1 \in [0;1/2[$, $f_M \in X^{b_1}_{\lambda,M}$, and $\gamma \in L^2(\mathbb{R})$ satisfying*

$$|\widehat{\gamma(\tau)}| \lesssim \langle\tau\rangle^{-4} \tag{3.4.9}$$

*Then, for any $K_0 \geqslant 1$ and $t_0 \in \mathbb{R}$ :*

$$K_0^{1/2}\beta^{b_1}_{M,K_0}\left|\left|\chi_{K_0}(\tau-\omega)\mathcal{F}\left\{\gamma(K_0(t-t_0))\mathcal{F}^{-1}f_M\right\}\right|\right|_{L^2} \lesssim \beta^{b_1}_{M,K_0}\,||f_M||_{X^0_{\lambda,M}} \tag{3.4.10}$$

*and*

$$\sum_{K\geqslant K_0} K^{1/2}\beta^{b_1}_{M,K}\left|\left|\rho_K(\tau-\omega)\mathcal{F}\left\{\gamma(K_0(t-t_0))\mathcal{F}^{-1}f_M\right\}\right|\right|_{L^2} \lesssim ||f_M||_{X^{b_1}_{\lambda,M}} \tag{3.4.11}$$

*and the implicit constants are independent of $K_0$, $t_0$, $M$ or $\lambda$.*

We will have several uses of the following estimate

**Lemma 3.4.2**

*For any $M \in 2^{\mathbb{Z}}$ and $f_M \in X^0_{\lambda,M}$, we have*

$$||f_M||_{L^2_{\xi,q}L^1_\tau} \lesssim ||f_M||_{X^0_{\lambda,M}} \tag{3.4.12}$$

**Proof :**
We decompose $f_M$ according to its modulations :

$$||f_M||_{L^2_{\xi,q}L^1_\tau} \leqslant \sum_{K\geqslant 1} ||\rho_K(\tau-\omega)f_M||_{L^2_{\xi,q}L^1_\tau}$$
$$\lesssim \sum_{K\geqslant 1} K^{1/2} \left|\left|\widetilde{\rho_K}(\tau-\omega)\langle\tau-\omega\rangle^{-1/2}\cdot\rho_K(\tau-\omega)f_M\right|\right|_{L^2_{\xi,q}L^1_\tau}$$

Next, using Cauchy-Schwarz inequality in the $\tau$ variable, we control the previous term with

$$\sum_{K\geqslant 1} K^{1/2} \left|\left|\left|\left|\widetilde{\rho_K}(\tau-\omega)\langle\tau-\omega\rangle^{-1/2}\right|\right|_{L^2_\tau} ||\rho_K(\tau-\omega)f_M||_{L^2_\tau}\right|\right|_{L^2_{\xi,q}}$$

Now, since for any fixed $(\xi,q)\in\mathbb{R}\times\lambda^{-1}\mathbb{Z}$, $\left|\left|\widetilde{\rho_K}(\tau-\omega)\langle\tau-\omega\rangle^{-1/2}\right|\right|_{L^2_\tau}\lesssim 1$, the sum above is finally estimated by

$$\sum_{K\geqslant 1} K^{1/2}\,||\rho_K(\tau-\omega)f_M||_{L^2_{\xi,q,\tau}} = ||f_M||_{X^0_{\lambda,M}}$$

□

Now we prove the proposition.
**Proof :**
Let us begin by proving (3.4.10). Using that $||\chi_{K_0}(\tau-\omega)||_{L^2}\lesssim K_0^{1/2}$, we estimate the term on the left-hand side by

$$K_0^{1/2}\beta^{b_1}_{M,K_0}\left|\left|\,||\chi_{K_0}(\tau-\omega)||_{L^2_\tau}\left|\left|\left(K_0^{-1}\mathrm{e}^{i\tau' t_0}\widehat{\gamma}(K_0^{-1}\tau')\right)\star_\tau f_M\right|\right|_{L^\infty_\tau}\right|\right|_{L^2_{\xi,q}}$$
$$\lesssim \beta^{b_1}_{M,K_0}\left|\left|\left|\left|\left(\mathrm{e}^{i\tau' t_0}\widehat{\gamma}(K_0^{-1}\tau')\right)\star_\tau f_M\right|\right|_{L^\infty_\tau}\right|\right|_{L^2_{\xi,q}}$$

(3.4.10) then follows from using Young's inequality $L^\infty\times L^1\to L^\infty$ and (3.4.12), since $\widehat{\gamma}\in L^\infty$ by the assumption (3.4.9).

Now we prove (3.4.11). We decompose $f_M$ according to its modulations and then distinguish two cases depending on the relation between $K$ and $K_1$ :

$$\sum_{K\geqslant K_0} K^{1/2}\beta^{b_1}_{M,K}\left|\left|\rho_K(\tau-\omega)\mathcal{F}\left\{\gamma(K_0(t-t_0))\mathcal{F}^{-1}f_M\right\}\right|\right|_{L^2}$$
$$\leqslant \sum_{K\geqslant K_0} K^{1/2}\beta^{b_1}_{M,K}\sum_{K_1\geqslant 1}\left|\left|\rho_K(\tau-\omega)\left(\mathrm{e}^{i\tau' t_0}\widehat{\gamma_{K_0^{-1}}}\right)\star_\tau\left(\rho_{K_1}(\tau-\omega)f_M\right)\right|\right|_{L^2}$$
$$=\sum_{K\geqslant K_0}\sum_{K_1\leqslant K/10}()+\sum_{K\geqslant K_0}\sum_{K_1\gtrsim K}()=I+II$$

For the first term, we have $|\tau-\tau'|\sim K$ since $|\tau-\omega|\sim K$ and $|\tau'-\omega|\sim K_1\leqslant K/10$, thus using Young inequality $L^\infty\times L^1\to L^\infty$, the estimate $||\rho_K||_{L^2}\lesssim K^{1/2}$ and then summing on $K\geqslant K_0$, we get the bound

$$I \lesssim \sum_{K\geqslant K_0} K^{-1}\beta_{M,K}^{b_1} \sum_{K_1\leqslant K/10} \left|\left|\rho_K(\tau-\omega)\left(|\tau'|^{3/2}\widehat{\gamma_{K_0^{-1}}}\right)\star_\tau(\rho_{K_1}(\tau-\omega)f_M)\right|\right|_{L^2}$$

$$\lesssim K_0^{-1/2} \sum_{K_1\leqslant K/10} \left|\left||\tau'|^{3/2}\widehat{\gamma_{K_0^{-1}}}(\tau')\right|\right|_{L^\infty} ||\rho_{K_1}(\tau-\omega)f_M||_{L^2_{\xi,q}L^1_\tau}$$

This is enough for (3.4.11) after using (3.4.12) and

$$\left|\left||\cdot|^s\widehat{\gamma_{K_0^{-1}}}\right|\right|_{L^p} \lesssim K_0^{s+1/p-1}\,|||\cdot|^s\widehat{\gamma}||_{L^p} \tag{3.4.13}$$

and the right-hand side is finite by the assumption on gamma (3.4.9).

Finally, $II$ is simply controlled using Young $L^1\times L^2\to L^2$ and (3.4.13) :

$$II \lesssim \sum_{K_1\gtrsim K_0} K_1^{1/2}\beta_{M,K_1}^{b_1}\left|\left|\widehat{\gamma_{K_0^{-1}}}\right|\right|_{L^1} ||\rho_{K_1}(\tau-\omega)f_M||_{L^2} \lesssim ||f_M||_{X^{b_1}_{\lambda,M}}$$

□

**Remark 3.4.3.** *For the loss in (3.4.10) to be trivial, we need either $b_1=0$ or $K_0\lesssim(1\vee M)^3$. In particular, in the multilinear estimates we cannot localize the term with the smallest frequency on time intervals of size $M_{max}^{-1}$ when $b_1>0$.*

The next proposition deals with general time multipliers as in [IKT08] :

**Proposition 3.4.4**

*Let $M>0$, $b_1\in[0;1/2[$, $f_M\in F^{b_1}_{\lambda,M}$ (repectively $N^{b_1}_{\lambda,M}$) and $m_M\in\mathcal{C}^4(\mathbb{R})$ bounded along with its derivatives. Then*

$$||m_M(t)f_M||_{F^{b_1}_{\lambda,M}} \lesssim \left(\sum_{k=0}^{4}(1\vee M)^{-k}\left|\left|m_M^{(k)}\right|\right|_{L^\infty}\right)||f_M||_{F^{b_1}_{\lambda,M}} \tag{3.4.14}$$

*and*

$$||m_M(t)f_M||_{N^{b_1}_{\lambda,M}} \lesssim \left(\sum_{k=0}^{4}(1\vee M)^{-k}\left|\left|m_M^{(k)}\right|\right|_{L^\infty}\right)||f_M||_{N^{b_1}_{\lambda,M}} \tag{3.4.15}$$

*respectively, uniformly in $M>0$ and $\lambda\geqslant1$.*

**Proof :**
Using the definition of $F^{b_1}_{\lambda,M}$, we write

$$||m_Mf_M||_{F^{b_1}_{\lambda,M}} = \sup_{t_M\in\mathbb{R}}\sum_{K\geqslant1}K^{1/2}\beta^{b_1}_{M,K}\left|\left|p\cdot\rho_K(\tau-\omega)\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)m_M(t)f_M\right\}\right|\right|_{L^2}$$

Next we estimate

$$\left|\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)m_M\right\}\right|(\tau) \leqslant \left|\left|\chi_{(1\vee M)^{-1}}(t-t_M)m_M\right|\right|_{L^1} \lesssim (1\vee M)^{-1}\,||m_M||_{L^\infty}$$

and

$$\left|\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)m_M\right\}\right|(\tau)=|\tau|^{-4}\left|\mathcal{F}\frac{\mathrm{d}^4}{\mathrm{d}t^4}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)m_M\right\}\right|$$
$$\lesssim|\tau|^{-4}\sum_{k=0}^{4}\left|\left|m_M^{(k)}\right|\right|_{L^\infty}(1\vee M)^{3-k}\left|\left|\chi^{(4-k)}\right|\right|_{L^1}$$

Thus we obtain

$$\left|\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)m_M\right\}\right|(\tau)$$
$$\lesssim\left(\sum_{k=0}^{4}(1\vee M)^{-k}\left|\left|m_M^{(k)}\right|\right|_{L^\infty}\right)(1\vee M)^{-1}\left\langle(1\vee M)^{-1}\tau\right\rangle^{-4}$$

Using (3.4.10) and (3.4.11) with $t_0=t_M$, $K_0=(1\vee M)$ and $\gamma(t)=\mathcal{F}^{-1}\langle\tau\rangle^{-4}$ concludes the proof of (3.4.14). The proof of (3.4.15) follows similarly.

□

The last proposition justifies the use of $\mathbf{F}_\lambda(T)$ as a resolution space :

**Proposition 3.4.5**

*Let $\alpha\in\mathbb{N}^*$, $T\in]0;1]$, $b_1\in[0;1/2[$ and $u\in\mathbf{F}_\lambda^{\alpha,b_1}(T)$. Then*

$$||u||_{L_T^\infty\mathbf{E}_\lambda^\alpha}\lesssim||u||_{\mathbf{F}_\lambda^{\alpha,b_1}(T)}\tag{3.4.16}$$

*and*

$$||u||_{L_T^\infty L_{xy}^2}\lesssim||u||_{\overline{\mathbf{F}_\lambda}^{b_1}(T)}\tag{3.4.17}$$

**Proof :**

The proof is the same as in [IKT08, Lemma 3.1] : let $M\in2^{\mathbb{Z}}$, $\widetilde{u_M}$ be an extension of $P_Mu$ to $\mathbb{R}$ with $||\widetilde{u_M}||_{F_{\lambda,M}^{b_1}}\leqslant2\,||P_Mu||_{F_{\lambda,M}^{b_1}(T)}$ and $t_M\in[-T;T]$, then it suffices to prove that

$$||p\cdot\mathcal{F}_{xy}\widetilde{u_M}(t_M)||_{L_{\xi,q}^2}\lesssim\left|\left|p\cdot\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)\widetilde{u_M}\right\}\right|\right|_{X_{\lambda,M}^{b_1}}$$

Using the properties of $\chi$ and the inversion formula, we can write

$$\widetilde{u_M}(t_M)=\left\{\chi_{(1\vee M)^{-1}}(\cdot-t_M)\widetilde{u_M}\right\}(t_M)=\int_{\mathbb{R}}\mathcal{F}_t\left\{\chi_{(1\vee M)^{-1}}(t-t_M)\widetilde{u_M}\right\}(\tau)\mathrm{e}^{it_M\tau}\mathrm{d}\tau$$

Thus, using (3.4.12), we get the final bound

$$||p\cdot\mathcal{F}_{xy}\widetilde{u_M}(t_M)||_{L_{\xi,q}^2}\leqslant\left|\left|p\cdot\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)\widetilde{u_M}\right\}\right|\right|_{L_{\xi,q}^2L_\tau^1}$$
$$\lesssim\left|\left|p\cdot\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t-t_M)\widetilde{u_M}\right\}\right|\right|_{X_{\lambda,M}^{b_1}}$$

□

## 3.5 Linear estimates

This section deals with (3.1.11) and (3.1.14).

**Proposition 3.5.1**

*Let $T>0$, $b_1\in[0;1/2[$ and $u\in\mathbf{B}^{\alpha}_{\lambda}(T)$, $f\in\mathbf{N}^{\alpha,b_1}_{\lambda}(T)$ satisfying*

$$\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u=f \tag{3.5.1}$$

*on $[-T,T]\times\mathbb{R}\times\mathbb{T}_\lambda$.*
*Then $u\in\mathbf{F}^{\alpha,b_1}_{\lambda}(T)$ and*

$$||u||_{\mathbf{F}^{\alpha,b_1}_{\lambda}(T)}\lesssim||u||_{\mathbf{B}^{\alpha}_{\lambda}(T)}+||f||_{\mathbf{N}^{\alpha,b_1}_{\lambda}(T)} \tag{3.5.2}$$

**Proof :**
This proposition is proved in [IKT08] (see also [KP15]). We recall the proof here for completeness.

Looking at the definition of $\mathbf{F}^{\alpha,b_1}_{\lambda}(T)$ (3.4.2), $\mathbf{N}^{\alpha,b_1}_{\lambda}(T)$ (3.4.3) and $\mathbf{B}^{\alpha}_{\lambda}(T)$ (3.4.4), we have to prove that $\forall M>0$,

$$||P_M u||_{F^{b_1}_{\lambda,M}(T)}\lesssim||P_M u_0||_{\mathbf{E}^0_\lambda}+||P_M f||_{N^{b_1}_{\lambda,M}(T)}\ \text{ if } 0<M\leqslant 1$$
$$||P_M u||_{F^{b_1}_{\lambda,M}(T)}\lesssim\sup_{t_M\in[-T,T]}||P_M u(t_M)||_{\mathbf{E}^0_\lambda}+||P_M f||_{N^{b_1}_{\lambda,M}(T)}\ \text{ if } M>1$$

Let $M>0$. As in [KP15, Proposition 2.9, p.14], we begin by constructing extensions $\widetilde{u}_M$ (respectively $\widetilde{f}_M$) of $P_M u$ (respectively $P_M f$) to $\mathbb{R}$, with a control on the boundary terms.
To do so, we first define the smooth cutoff function

$$m_M(t):=\begin{cases}\chi_{(1\vee M)^{-1}/10}(t+T)\text{ if } t<-T\\ 1\text{ if } t\in[-T,T]\\ \chi_{(1\vee M)^{-1}/10}(t-T)\text{ if } t>T\end{cases}$$

Next, we define $\widetilde{f}_M$ on $\mathbb{R}$ with

$$\widetilde{f_M}(t):=m_M(t)f_M(t) \tag{3.5.3}$$

where $f_M$ is an extension of $P_M f$ to $\mathbb{R}$ satisfying $||f_M||_{N^{b_1}_{\lambda,M}}\leqslant 2\,||P_M f||_{N^{b_1}_{\lambda,M}(T)}$.

So $\widetilde{f_M}$ is also an extension of $P_M f$, with $\text{supp}\widetilde{f_M}\subset[-T-(1\vee M)^{-1}/5,T+(1\vee M)^{-1}/5]$.
From (3.5.1), we have that

$$P_M u(t)=U(t)P_M u_0+\int_0^t U(t-t')P_M f(t')\mathrm{d}t'\text{ on }[-T,T] \tag{3.5.4}$$

Thus we define $\widetilde{u_M}$ as

$$\widetilde{u_M}(t):=m_M(t)\left\{U(t)P_M u_0+\int_0^t U(t-t')\widetilde{f_M}(t')\mathrm{d}t'\right\},\ t\in\mathbb{R} \tag{3.5.5}$$

The choice of $\widetilde{f_M}$ and $\widetilde{u_M}$ is dictated from the necessity to control the boundary term. First using (3.4.15) with $m_M$ we have

$$\left|\left|\widetilde{f_M}\right|\right|_{N^{b_1}_{\lambda,M}}\lesssim||P_M f||_{N^{b_1}_{\lambda,M}(T)}$$

and $\widetilde{u_M}$ defines an extension of $P_M u$.
Moreover, if $t_M \notin [-T,T]$, from the choice of $m_M$, we can write $\chi_{(1\vee M)^{-1}}(t-t_M)\widetilde{u_M}(t) = \chi_{(1\vee M)^{-1}}(t-\widetilde{t_M})\chi_{(1\vee M)^{-1}}(t-t_M)\widetilde{u_M}(t)$ for a $\widetilde{t_M} \in [-T,T]$. Then, using (3.4.10) and (3.4.11) we get

$$\sup_{t_M \notin [-T,T]} \left|\left|\chi_{(1\vee M)^{-1}}(t-t_M)\widetilde{u_M}\right|\right|_{X^{b_1}_{\lambda,M}} \lesssim \sup_{\widetilde{t_M}\in[-T,T]} \left|\left|\chi_{(1\vee M)^{-1}}(t-\widetilde{t_M})\widetilde{u_M}\right|\right|_{X^{b_1}_{\lambda,M}}$$

Thus it suffices to prove

$$\sup_{t_M\in[-T,T]} ||p\cdot\mathcal{F}\left\{\chi(t-t_M)\widetilde{u_M}\right\}||_{X^{b_1}_{\lambda,M}} \lesssim ||\widetilde{u_M}(0)||_{\mathbf{E}^0_\lambda} + \sup_{\widetilde{t_M}\in\mathbb{R}} \left|\left|(\tau-\omega+i)^{-1}p\cdot\mathcal{F}\left\{\chi(t-\widetilde{t_M})\widetilde{f_M}\right\}\right|\right|_{X^{b_1}_{\lambda,M}} \quad \text{if } M\leqslant 1$$

and

$$\sup_{t_M\in[-T,T]} ||p\cdot\mathcal{F}\left\{\chi_{M^{-1}}(t-t_M)\widetilde{u_M}\right\}||_{X^{b_1}_{\lambda,M}} \lesssim \sup_{\widehat{t_M}\in[-T,T]} \left|\left|\widetilde{u_M}(\widehat{t_M})\right|\right|_{\mathbf{E}^0_\lambda} + \sup_{\widetilde{t_M}\in\mathbb{R}} \left|\left|(\tau-\omega+iM)^{-1}p\cdot\mathcal{F}\left\{\chi_{M^{-1}}(t-\widetilde{t_M})\widetilde{f_M}\right\}\right|\right|_{X^{b_1}_{\lambda,M}} \quad \text{if } M>1$$

Note that, since $m_M \equiv 1$ on $[-T,T]$ and $u$ is a solution of (3.5.1), for $t_M \in [-T,T]$, we have

$$P_M u(t_M) = U(t_M)P_M u_0 + \int_0^{t_M} U(t-t')\widetilde{f_M}(t')\mathrm{d}t'$$

and thus

$$\widetilde{u_M}(t+t_M) = m_M(t+t_M)\left\{U(t)P_M u(t_M) + \int_0^t U(t-t')\widetilde{f_M}(t'+t_M)\mathrm{d}t'\right\}$$

Finally, it suffices to prove that

$$||p\cdot\mathcal{F}\left\{\chi(t-t_M)m_M(t)U(t)P_M u_0\right\}||_{X^{b_1}_{\lambda,M}} \lesssim ||\widetilde{u_M}(0)||_{\mathbf{E}^0_\lambda} \tag{3.5.6}$$

and

$$\left|\left|p\cdot\mathcal{F}\left\{\chi(t-t_M)m_M(t)\int_0^t U(t-t')\widetilde{f_M}(t')\mathrm{d}t'\right\}\right|\right|_{X^{b_1}_{\lambda,M}} \lesssim \left|\left|(\tau-\omega+i)^{-1}p\cdot\mathcal{F}\left\{\chi(t-t_M)\widetilde{f_M}\right\}\right|\right|_{X^{b_1}_{\lambda,M}} \tag{3.5.7}$$

for the low-frequency part, and

$$||p\cdot\mathcal{F}\left\{\chi_{M^{-1}}(t)m_M(t+t_M)U(t)P_M u(t_M)\right\}||_{X^{b_1}_{\lambda,M}} \lesssim ||\widetilde{u_M}(t_M)||_{\mathbf{E}^0_\lambda} \tag{3.5.8}$$

and

$$\left|\left|p\cdot\mathcal{F}\left\{\chi_{M^{-1}}(t)m_M(t+t_M)\int_0^t U(t-t')\widetilde{f_M}(t_M+t')\mathrm{d}t'\right\}\right|\right|_{X^{b_1}_{\lambda,M}} \lesssim \left|\left|(\tau-\omega+iM)^{-1}p\cdot\mathcal{F}\left\{\chi_{M^{-1}}(t-t_M)\widetilde{f_M}\right\}\right|\right|_{X^{b_1}_{\lambda,M}} \tag{3.5.9}$$

for the high-frequency part.
To prove those estimates, we first notice that, since $t' \in [0;t]$ and $t \in \operatorname{supp}\chi_{(1\vee M)^{-1}}$, we can write $\widetilde{f_M}$ as

$$\widetilde{f_M}(t_M + t') = \sum_{|n|\leqslant 100} f_{M,n}(t_M + t') := \sum_{|n|\leqslant 100} \gamma\left((1\vee M)t' - n\right)\widetilde{f_M}(t_M + t')$$

where $\gamma : \mathbb{R} \to [0;1]$ is a smooth partition of unity, satisfying $\operatorname{supp}\gamma \subset [-1;1]$ and for all $x \in \mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} \gamma(x-n) = 1$$

The second observation is that, for a fixed $t_M$, we have for the homogeneous term

$$||p \cdot \mathcal{F}\left\{\chi_{M^{-1}}(t) m_M(t+t_M) U(t) P_M u(t_M)\right\}||_{X^{b_1}_{\lambda,M}} \lesssim ||m_M U(t) P_M u(t_M)||_{F^{b_1}_{\lambda,M}}$$

so we can remove the localization $m_M(t)$ thanks to (3.4.14), and similarly for the inhomogeneous term.
Computing the Fourier transform in the left-hand side of (3.5.6) and using the bound

$$\left|\left|\rho_K(\tau-\omega)\mathrm{e}^{it_M(\tau-\omega)}\widehat{\chi}(\tau-\omega)\right|\right|_{L^2_\tau} \lesssim \left|\left|\rho_K(\tau)\langle\tau\rangle^{-2}\right|\right|_{L^2} \lesssim K^{-3/2}$$

since $\widehat{\chi} \in \mathcal{S}(\mathbb{R})$, we then obtain

$$\begin{aligned}||p \cdot \mathcal{F}\left\{\chi(t-t_M)U(t)P_M u_0\right\}||_{X^{b_1}_{\lambda,M}} \\ \lesssim \sum_{K\geqslant 1} K^{1/2}\beta^{b_1}_{M,K}\left|\left|\rho_K(\tau-\omega)p\cdot \mathrm{e}^{it_M(\tau-\omega)}\widehat{\chi}(\tau-\omega)\widehat{P_M u_0}\right|\right|_{L^2} \lesssim ||P_M u_0||_{L^2}\end{aligned}$$

The proof of (3.5.8) is the same replacing the first bound by

$$\left|\left|\rho_K(\tau-\omega)M^{-1}\left\langle M^{-1}(\tau-\omega)\right\rangle^{-2}\right|\right|_{L^2_\tau} \lesssim M^{-1}K^{1/2}(1\vee M^{-1}K)^{-2}$$

For (3.5.7) and (3.5.9), a computation gives first

$$\begin{aligned}&\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(t)\int_0^t U(t-t')f_{M,n}(t_M+t')\mathrm{d}t'\right\}(\tau) \\ &\quad = (1\vee M)^{-1}\int_{\mathbb{R}} \frac{\widehat{\chi}((1\vee M)^{-1}(\tau-\tau')) - \widehat{\chi}((1\vee M)^{-1}(\tau-\omega))}{i(\tau'-\omega)} \\ &\qquad\qquad \cdot \mathrm{e}^{it_M\tau'}\mathcal{F}\left\{f_{M,n}\right\}(\tau')\mathrm{d}\tau'\end{aligned}$$

Now, we distinguish between two cases, whether $|\tau'-\omega+i(1\vee M)| \sim |\tau'-\omega|$ or $|\tau'-\omega+i(1\vee M)| \sim (1\vee M)$.
First, if $|\tau'-\omega| \gtrsim (1\vee M)$, we have

$$\begin{aligned}&\left|\frac{\widehat{\chi}((1\vee M)^{-1}(\tau-\tau')) - \widehat{\chi}((1\vee M)^{-1}(\tau-\omega))}{i(\tau'-\omega)}\right| \\ &\qquad \lesssim \left|\frac{\widehat{\chi}((1\vee M)^{-1}(\tau-\tau'))}{i(\tau'-\omega+i(1\vee M))}\right| + \left|\frac{\widehat{\chi}((1\vee M)^{-1}(\tau-\omega))}{i(\tau'-\omega+i(1\vee M))}\right|\end{aligned}$$

Now if $|\tau'-\omega|\lesssim(1\vee M)$ we apply the mean value theorem to $\widehat{\chi}$ so that

$$\widehat{\chi}((1\vee M)^{-1}(\tau-\tau'))-\widehat{\chi}((1\vee M)^{-1}(\tau-\omega))=(1\vee M)^{-1}\widehat{\chi}'(\theta)\cdot(\tau'-\omega)$$

for a $\theta\in[\tau-\tau';\tau-\omega]$. Thus we have

$$\begin{aligned}\left|\frac{\widehat{\chi}((1\vee M)^{-1}(\tau-\tau'))-\widehat{\chi}((1\vee M)^{-1}(\tau-\omega))}{i(\tau'-\omega)}\right| &\lesssim (1\vee M)^{-1}\,|\widehat{\chi}'(\theta)| \\ &\lesssim |\tau'-\omega+i(1\vee M)|^{-1}|\widehat{\chi}'(\theta)|\end{aligned}$$

Finally, using the assumption on $\theta$ and that $\widehat{\chi}\in\mathcal{S}(\mathbb{R})$, we have in both cases

$$\begin{aligned}&\left|\frac{\widehat{\chi}((1\vee M)^{-1}(\tau-\tau'))-\widehat{\chi}((1\vee M)^{-1}(\tau-\omega))}{i(\tau'-\omega)}\right| \\ &\qquad\lesssim\left|\frac{\big\langle(1\vee M)^{-1}(\tau-\tau')\big\rangle^{-4}}{\tau'-\omega+i(1\vee M)}\right|+\left|\frac{\big\langle(1\vee M)^{-1}(\tau-\omega)\big\rangle^{-4}}{\tau'-\omega+i(1\vee M)}\right|\end{aligned}$$

Coming back to (3.5.7) and (3.5.9), the left-hand side can be split into

$$\begin{aligned}&\sum_{|n|\leqslant 100}\left|\left|p\cdot(1\vee M)^{-1}\int_{\mathbb{R}}\left|\frac{\big\langle(1\vee M)^{-1}(\tau-\tau')\big\rangle^{-4}}{\tau'-\omega+i(1\vee M)}\mathcal{F}\left\{f_{M,n}\right\}(\tau')\right|\mathrm{d}\tau'\right|\right|_{X^{b_1}_{\lambda,M}} \\ &\qquad+\sum_{|n|\leqslant 100}\left|\left|p\cdot(1\vee M)^{-1}\int_{\mathbb{R}}\left|\frac{\big\langle(1\vee M)^{-1}(\tau-\omega)\big\rangle^{-4}}{\tau'-\omega+i(1\vee M)}\mathcal{F}\left\{f_{M,n}\right\}(\tau')\right|\mathrm{d}\tau'\right|\right|_{X^{b_1}_{\lambda,M}}\end{aligned}$$

The first term is handled with (3.4.10) and (3.4.11) with $K_0=(1\vee M)$ and $\gamma=\mathcal{F}^{-1}\left\{\langle\cdot\rangle^{-4}\right\}$. This term is thus controlled by

$$\sup_{|n|\leqslant 100}\left|\left|p\cdot(\tau-\omega+i(1\vee M))^{-1}\mathcal{F}\left\{f_{M,n}\right\}\right|\right|_{X^{b_1}_{\lambda,M}}\lesssim\left|\left|\widetilde{f_M}\right|\right|_{F^{b_1}_{\lambda,M}}$$

where in the last step we have used that

$$\gamma((1\vee M)t-n)=\gamma((1\vee M)t-n)\chi_{(1\vee M)^{-1}}(t-(1\vee M)^{-1}n)$$

and (3.4.10)-(3.4.11) to get rid of $\gamma$.

It remains to treat the second term. By definition of the $X^{b_1}_{\lambda,M}$ norm, we can write it

$$\begin{aligned}&\sum_{K\geqslant 1}K^{1/2}\beta^{b_1}_{M,K}\,\bigg|\bigg|\rho_K(\tau-\omega)p\cdot(1\vee M)^{-1} \\ &\qquad\cdot\int_{\mathbb{R}}\left|\frac{\big\langle(1\vee M)^{-1}(\tau-\omega)\big\rangle^{-4}}{\tau'-\omega+i(1\vee M)}\mathcal{F}\left\{f_{M,n}\right\}(\tau')\right|\mathrm{d}\tau'\bigg|\bigg|_{L^2} \\ &=\sum_{K\geqslant 1}K^{1/2}\beta^{b_1}_{M,K}\,\bigg|\bigg|p\cdot\left|\left|(\tau'-\omega+i(1\vee M))^{-1}\mathcal{F}\left\{f_{M,n}\right\}\right|\right|_{L^1_{\tau'}} \\ &\qquad\cdot\left|\left|\rho_K(\tau-\omega)(1\vee M)^{-1}\big\langle(1\vee M)^{-1}(\tau-\omega)\big\rangle^{-4}\right|\right|_{L^2_\tau}\bigg|\bigg|_{L^2_{\xi,q}}\end{aligned}$$

Now, since

$$\sum_{K\geqslant 1} K^{1/2}\beta_{M,K}^{b_1}(1\vee M)^{-1}\langle(1\vee M)^{-1}K\rangle^{-4}||\rho_K||_{L^2}\lesssim 1$$

we can use (3.4.12) to bound the last term with

$$\Big|\Big|p\cdot||(\tau'-\omega+i(1\vee M))^{-1}\mathcal{F}\{f_{M,n}\}||_{L^1_{\tau'}}\Big|\Big|_{L^2_{\xi,q}}\lesssim||p\cdot(\tau'-\omega+i(1\vee M))^{-1}\mathcal{F}\{f_{M,n}\}||_{X^{b_1}_{\lambda,M}}$$

which concludes the proof through the same argument than above.

□

Proceeding in the same way at the $L^2$ level, we have also

**Proposition 3.5.2**

*Let $T>0$, $b_1\in[0;1/2[$ and $u\in\overline{\mathbf{B}_\lambda}^{b_1}(T)$, $f\in\overline{\mathbf{N}_\lambda}^{b_1}(T)$ satisfying (3.5.1) on $[-T,T]\times\mathbb{R}\times\mathbb{T}_\lambda$. Then*

$$||u||_{\overline{\mathbf{F}_\lambda}^{b_1}(T)}\lesssim||u||_{\overline{\mathbf{B}_\lambda}(T)}+||f||_{\overline{\mathbf{N}_\lambda}^{b_1}(T)} \tag{3.5.10}$$

## 3.6 Dyadic estimates

As in the standard Bourgain method, we will need some bilinear estimates for functions localized in both their frequency and their modulation. This section deals with estimating expressions under the form $\int f_1\star f_2\cdot f_3$ which will be useful to prove the main bilinear estimate (3.1.12) as well as the energy estimate (3.1.13). The following lemma gives a first rough estimate :

**Lemma 3.6.1**

*Let $f_i\in L^2(\mathbb{R}^2\times\lambda^{-1}\mathbb{Z})$ be such that $\text{supp}f_i\subset D_{\lambda,M_i,\leqslant K_i}\cap\mathbb{R}^2\times I_i$, with $M_i\in 2^{\mathbb{Z}}$, $K_i\in 2^{\mathbb{N}}$ and $I_i\subset\lambda^{-1}\mathbb{Z}$, $i=1,2,3$. Then*

$$\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_1\star f_2\cdot f_3\lesssim M_{min}^{1/2}K_{min}^{1/2}|I|_{min}^{1/2}\prod_{i=1}^{3}||f_i||_{L^2} \tag{3.6.1}$$

**Proof :**
The proof is the same as in [IKT08, Lemma 5.1 (b)]. We just have to expand the convolution product in the left-hand side and then apply Cauchy-Schwarz inequality in the variable coresponding to the min : if, for example, $K_1=K_{min}$, we have

$$\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_1\star f_2\cdot f_3=\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}}\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_1(\tau-\tau_2,\zeta-\zeta_2)\cdot f_2(\tau_2,\zeta_2)f_3(\tau,\zeta)\mathrm{d}\tau_2\mathrm{d}\tau\mathrm{d}\zeta_2\mathrm{d}\zeta$$

Using Cauchy-Schwarz inequality in $\tau$, the previous term is less than

$$\int_{\mathbb{R}\times\lambda^{-1}\mathbb{Z}}\int_{\mathbb{R}\times\lambda^{-1}\mathbb{Z}}||f_3(\zeta)||_{L^2_\tau}\Big|\Big|\int_{\mathbb{R}} f_1(\tau-\tau_2,\zeta-\zeta_2)f_2(\tau_2,\zeta_2)\mathrm{d}\tau_2\Big|\Big|_{L^2_\tau}\mathrm{d}\zeta_2\mathrm{d}\zeta$$

Next, a use of Young's inequality $L^1 \times L^2 \to L^2$ in $\tau$ gives the bound

$$\int_{\mathbb{R}\times\lambda^{-1}\mathbb{Z}} \int_{\mathbb{R}\times\lambda^{-1}\mathbb{Z}} ||f_3(\zeta)||_{L^2_\tau} \, ||f_2(\zeta_2)||_{L^2_\tau} \, ||f_1(\zeta-\zeta_2)||_{L^1_{\tau_1}} \, \mathrm{d}\zeta_2 \mathrm{d}\zeta$$

Finally, using again Cauchy-Schwarz inequality in $\tau_1$, the previous term is controlled with

$$\int_{\mathbb{R}\times\lambda^{-1}\mathbb{Z}} \int_{\mathbb{R}\times\lambda^{-1}\mathbb{Z}} ||f_3(\zeta)||_{L^2_\tau} \, ||f_2(\zeta_2)||_{L^2_\tau} \, K_1^{1/2} \, ||f_1(\zeta-\zeta_2)||_{L^2_{\tau_1}} \, \mathrm{d}\zeta_2 \mathrm{d}\zeta$$

We get (3.6.1) when proceeding similarly for the integrals in $\xi$ and $q$.

□

### 3.6.1 Localized Strichartz estimates

The purpose of this subsection is to improve (3.6.1). All the estimates we need are already used in [MST11] in the context of the KP-II equation. We briefly recall the outline of the proof here for the sake of completeness.

First, we are going to use the following easy lemmas :

**Lemma 3.6.2**

*Let $\Lambda \subset \mathbb{R} \times \lambda^{-1}\mathbb{Z}$. We assume that the projection of $\Lambda$ on the $\xi$ axis is contained in an interval $I \subset \mathbb{R}$. Moreover, we assume that the measure of the $q$-sections of $\Lambda$ (that is the sets $\left\{q \in \lambda^{-1}\mathbb{Z}, (\xi_0, q) \in \Lambda\right\}$ for a fixed $\xi_0$) is uniformly (in $\xi_0$) bounded by a constant $C$. Then we have*

$$|\Lambda| \leqslant C|I| \tag{3.6.2}$$

**Proof :**
The proof is immediate : by definition

$$|\Lambda| = \int_I \left( \int \mathbb{1}_\Lambda(\xi, q)(\mathrm{d}q)_\lambda \right) \mathrm{d}\xi \leqslant \int_I C \mathrm{d}\xi = C\,|I|$$

□

**Lemma 3.6.3**

*Let $I$, $J$ be two intervals in $\mathbb{R}$, and let $\varphi : I \to \mathbb{R}$ be a $\mathcal{C}^1$ function with $\inf_{\xi \in J} |\varphi'(\xi)| > 0$. Then*

$$|\{x \in J, \ \varphi(x) \in I\}| \leqslant \frac{|I|}{\inf_{\xi\in J} |\varphi'(\xi)|} \tag{3.6.3}$$

*and*

$$\left|\left\{q \in J \cap \lambda^{-1}\mathbb{Z}, \ \varphi(q) \in I\right\}\right| \lesssim \left\langle \frac{|I|}{\inf_{\xi\in J} |\varphi'(\xi)|} \right\rangle \tag{3.6.4}$$

**Proof :**
Let us define

$$\mathcal{J} := \{x \in J, \ \varphi(x) \in I\}$$

and

$$\mathcal{J}_\lambda := \left\{q \in J \cap \lambda^{-1}\mathbb{Z}, \ \varphi(q) \in I\right\}$$

We just have to use the mean value theorem and write

$$|\mathcal{J}| = \sup_{x_1,x_2\in\mathcal{J}} |x_2 - x_1| = \sup_{x_1,x_2\in\mathcal{J}} \frac{|\varphi(x_2)-\varphi(x_1)|}{|\varphi'(\theta)|}$$

for a $\theta \in [x_1;x_2]$, and (3.6.3) follows since $\sup_{x_1,x_2\in\mathcal{J}} |\varphi(x_2)-\varphi(x_1)| \leqslant |I|$ by definition of $\mathcal{J}$. The proof of (3.6.4) is the same, using that

$$|\mathcal{J}_\lambda| \leqslant \lambda^{-1} + \sup_{q_1,q_2\in\mathcal{J}_\lambda} |q_2 - q_1|$$

by definition of $(\mathrm{d}q)_\lambda$.

□

**Lemma 3.6.4**

*Let $a \neq 0$ ,b, c be real numbers and $I \subset \mathbb{R}$ a bounded interval. Then*

$$\left|\left\{x\in\mathbb{R},\ ax^2+bx+c\in I\right\}\right| \lesssim \frac{|I|^{1/2}}{|a|^{1/2}} \tag{3.6.5}$$

*and*

$$\left|\left\{q\in\lambda^{-1}\mathbb{Z},\ aq^2+bq+c\in I\right\}\right| \lesssim \left\langle \frac{|I|^{1/2}}{|a|^{1/2}} \right\rangle \tag{3.6.6}$$

**Proof :**
We begin by proving (3.6.5). By the linear change of variable $x \mapsto x + b/(2a)$ it suffices to evaluate

$$\left|\left\{y\in\mathbb{R},\ ay^2\in\widetilde{I}\right\}\right| \text{ with } \widetilde{I} = I + b^2/(4a) - c,\ |\widetilde{I}| = |I|$$

Writing $\varepsilon := \mathrm{sign}(a)$, the measure of the previous set is

$$\int_{\mathbb{R}} \mathbb{1}_{\widetilde{I}}(ay^2)\mathrm{d}y = |a|^{-1/2}\int_{\mathbb{R}} \mathbb{1}_{\varepsilon\widetilde{I}}(x^2)\mathrm{d}x$$

— If $0 \notin \varepsilon\widetilde{I}$, by symmetry we may assume $\varepsilon\widetilde{I} \subset \mathbb{R}_+^*$ and write $\varepsilon\widetilde{I} = [x_1;x_2]$ with $0 < x_1 < x_2$. Then an easy computation gives

$$\begin{aligned}\left|\left\{y\in\mathbb{R},\ ay^2\in\widetilde{I}\right\}\right| &= |a|^{-1/2}\int_{\mathbb{R}} \mathbb{1}_{[x_1;x_2]}(x^2)\mathrm{d}x = |a|^{-1/2}\int_{\mathbb{R}} \mathbb{1}_{[x_1;x_2]}(y)\frac{\mathrm{d}y}{2\sqrt{y}} \\ &= |a|^{-1/2}\,[\sqrt{y}]_{x_1}^{x_2} = |a|^{-1/2}(\sqrt{x_2}-\sqrt{x_1}) \leqslant |a|^{-1/2}|I|^{1/2}\end{aligned}$$

— If $0 \in \varepsilon\widetilde{I}$ : defining $I^+ := (\varepsilon\widetilde{I} \cup -\varepsilon\widetilde{I}) \cap \mathbb{R}^+ = [0;x_2]$ we have

$$\left|\left\{y\in\mathbb{R},\ ay^2\in\widetilde{I}\right\}\right| \leqslant 2|a|^{-1/2}\int_{\mathbb{R}} \mathbb{1}_{I^+}(x^2)\mathrm{d}x = 2|a|^{-1/2}\sqrt{x_2} \lesssim |a|^{-1/2}|I|^{1/2}$$

The proof of (3.6.6) follows from (3.6.5) through the same argument as in the proof of (3.6.4).

□

The main estimates of this section are the following.

**Proposition 3.6.5** (***Dyadic $L^4 - L^2$ Strichartz estimate***)

*Let $M_1, M_2, M_3 \in 2^{\mathbb{Z}}$, $K_1, K_2, K_3 \in 2^{\mathbb{N}}$ and let $u_i \in L^2(\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z})$, $i = 1, 2$, be such that* $\mathrm{supp}(u_i) \subset D_{\lambda,M_i,\leqslant K_i}$. *Then*

$$\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot u_1 \star u_2\right|\right|_{L^2} \lesssim (K_1 \wedge K_2)^{1/2} M_{min}^{1/2} \cdot \left\langle (K_1 \vee K_2)^{1/4} (M_1 \wedge M_2)^{1/4} \right\rangle ||u_1||_{L^2} \, ||u_2||_{L^2} \quad (3.6.7)$$

*Moreover, if we are in the regime $K_{max} \leqslant 10^{-10} M_1 M_2 M_3$ then*

$$\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot u_1 \star u_2\right|\right|_{L^2} \lesssim (K_1 \wedge K_2)^{1/2} M_{min}^{1/2} \cdot \left\langle (K_1 \vee K_2)^{1/2} M_{max}^{-1/2} \right\rangle ||u_1||_{L^2} \, ||u_2||_{L^2} \quad (3.6.8)$$

**Proof :**

These estimates are proven in [MST11, Proposition 2.1 & Corollary 2.9] and [ST01, Theorem 2.1, p.456-458] for functions $f_i \in L^2(\mathbb{R}^2 \times \mathbb{Z})$ but with a slightly different support condition : the localization with respect to the modulations is done for the symbol of the linear operator associated with the KP-II equation (i.e $\widetilde{\omega}(\xi, q) = \xi^3 - q^2/\xi$), and the fifth-order KP-I equation ($\omega^{5\text{th}}(m, \eta) = -m^5 - \eta^2/m$) respectively. As a matter of fact, the proof only uses the form of the expression $(q_1/\xi_1 - q_2/\xi_2)$ but does not take into account its sign within the resonant function. Thus we can obtain the similar estimates for the KP-I equation. Let us recall the main steps in proving these estimates : first, split $u_1$ and $u_2$ depending on the value of $\xi_i$ on an $M_3$ scale

$$\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot u_1 \star u_2\right|\right|_{L^2} \leqslant \sum_{k \in \mathbb{Z}} \sum_{\ell \in \mathbb{Z}} \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot u_{1,k} \star u_{2,\ell}\right|\right|_{L^2}$$

with

$$u_{i,j} := \mathbb{1}_{[jM_3,(j+1)M_3]}(\xi_i) u_i$$

The conditions $|\xi| \sim M_3$, $\xi_1 \in [kM_3, (k+1)M_3]$ and $\xi - \xi_1 \in [\ell M_3; (\ell+1)M_3]$ require $\ell \in [-k - c; -k + c]$ for an absolute constant $c > 0$. Thus we have to get estimates for functions $u_i$ supported in $D_{\lambda,M_i,K_i} \cap \{\xi_i \in I_i\}$ for some intervals $I_i$.

Moreover, we may assume $\xi_i \geqslant 0$ on supp $u_i$ (see [ST01, p.460]). This is crucial as $\xi \sim \xi_1 \vee (\xi - \xi_1)$ in this case.

Squaring the left-hand side, it then suffices to evaluate

$$\int_{\mathbb{R}\times\mathbb{R}_+\times\lambda^{-1}\mathbb{Z}} \left| \int_{\mathbb{R}\times\mathbb{R}_+\times\lambda^{-1}\mathbb{Z}} \mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot u_1(\tau_1, \zeta_1) u_2(\tau - \tau_1, \zeta - \zeta_1) \mathrm{d}\tau_1 \mathrm{d}\xi_1 (\mathrm{d}q_1)_\lambda \right|^2 \mathrm{d}\tau \mathrm{d}\xi (\mathrm{d}q)_\lambda$$

Using Cauchy-Schwarz inequality, the integral above is controlled by

$$\sup_{\tau,\xi \geqslant 0, q \in D_{\lambda,M_3,\leqslant K_3}} |A_{\tau,\xi,q}| \cdot ||u_1||_{L^2}^2 \, ||u_2||_{L^2}^2$$

where $A_{\tau,\xi,q}$ is defined as

$$A_{\tau,\xi,q} := \big\{(\tau_1, \zeta_1) \in \mathbb{R} \times \mathbb{R}_+ \times \lambda^{-1}\mathbb{Z}, \xi_1 \in I_1, \xi - \xi_1 \in I_2,\ 0 \leqslant \xi_1 \sim M_1, \\ 0 \leqslant \xi - \xi_1 \sim M_2,\ \langle \tau_1 - \omega(\zeta_1) \rangle \lesssim K_1,\ \langle \tau - \tau_1 - \omega(\zeta - \zeta_1) \rangle \lesssim K_2\}$$

Using the triangle inequality in $\tau_1$, we get the bound

$$|A_{\tau,\xi,q}| \lesssim (K_1 \wedge K_2)\, |B_{\tau,\xi,q}|$$

where $B_{\tau,\xi,q}$ is defined as

$$\begin{aligned} B_{\tau,\xi,q} := \big\{ & \zeta_1 \in \mathbb{R}_+ \times \lambda^{-1}\mathbb{Z},\ \xi_1 \in I_1, \xi - \xi_1 \in I_2,\ 0 \leqslant \xi_1 \sim M_1, \\ & 0 \leqslant \xi - \xi_1 \sim M_2,\ \langle \tau - \omega(\zeta) - \Omega(\zeta_1, \zeta - \zeta_1, -\zeta) \rangle \lesssim (K_1 \vee K_2) \big\} \end{aligned}$$

where $\Omega$ is the resonant function for (3.1.6), defined on the hyperplane $\zeta_1 + \zeta_2 + \zeta_3 = 0$ :

$$\begin{aligned} \Omega(\zeta_1, \zeta_2, \zeta_3) := \omega(\zeta_1) + \omega(\zeta_2) + \omega(\zeta_3) &= -3\xi_1\xi_2\xi_3 + \frac{(\xi_1 q_2 - \xi_2 q_1)^2}{\xi_1\xi_2\xi_3} \\ &= -\frac{\xi_1\xi_2}{\xi_1 + \xi_2} \left\{ (\sqrt{3}\xi_1 + \sqrt{3}\xi_2)^2 - \left( \frac{q_1}{\xi_1} - \frac{q_2}{\xi_2} \right)^2 \right\} \end{aligned} \tag{3.6.9}$$

Now, (3.6.7) follows directly from applying lemma 3.6.2 and (3.6.6) to $B_{\tau,\xi,q}$ since its projection on the $\xi_1$ axis is controlled by $|I_1| \wedge |I_2|$, whereas for a fixed $\xi_1$, the cardinal of the $q_1$-section is estimated by $\Big\langle (K_1 \vee K_2)^{1/2} (M_1 \wedge M_2)^{1/2} \Big\rangle$ using (3.6.6) as $\tau - \omega(\zeta) - \Omega(\zeta_1, \zeta - \zeta_1, -\zeta)$ is a polynomial of second order in $q_1$, with a dominant coefficient $\sim (M_1 \wedge M_2)^{-1}$. Thus

$$|B_{\tau,\xi,q}| \lesssim (|I_1| \wedge |I_2|) \Big\langle (K_1 \vee K_2)^{1/2} (M_1 \wedge M_2)^{1/2} \Big\rangle$$

which gives the estimate (3.6.7) when applied with $I_1 = [kM_3; (k+1)M_3] \cap \mathfrak{I}_{M_1}$ and $I_2 = [\ell M_3, (\ell+1)M_3] \cap \mathfrak{I}_{M_2}$ and using Cauchy-Schwarz inequality to sum over $k \in \mathbb{Z}$.

In the case $K_{max} \leqslant 10^{-10} M_1 M_2 M_3$, we compute

$$\left| \frac{\partial \Omega}{\partial q_1} \right| = 2 \left| \frac{q_1}{\xi_1} - \frac{q - q_1}{\xi - \xi_1} \right| = 2 \left\{ \frac{\xi}{\xi_1(\xi - \xi_1)} \left( \Omega + 3\xi_1(\xi - \xi_1)\xi \right) \right\}^{1/2}$$

Thus, from the condition $|\Omega| \lesssim K_{max} \leqslant 10^{-10} M_1 M_2 M_3$ we get

$$\left| \frac{\partial \Omega}{\partial q_1} \right| \gtrsim \left| \frac{\xi}{\xi_1(\xi - \xi_1)} \cdot \xi_1(\xi - \xi_1)\xi \right|^{1/2} \sim M_{max}$$

At last, we can estimate $|B_{\tau,\xi,q}|$ in this regime by using (3.6.4) instead of (3.6.6), which gives the final bound

$$|B_{\tau,\xi,q}| \lesssim (|I_1| \wedge |I_2|) \langle (K_1 \vee K_2) M_{max}^{-1} \rangle$$

and (3.6.8) follows through the same argument as for (3.6.7).

□

**Remark 3.6.6.** *The estimate (3.6.7) is rather crude, yet sufficient for our purpose. (3.6.8) is better than (3.6.10) below in the regime $K_{max} \lesssim M_1 M_2 M_3$, $M_{min} \leqslant 1$. Thus we do not need to use some function spaces with a special low-frequency structure as in [IKT08] to deal with the difference equation, therefore we get a stronger uniqueness criterion. Note that we can perform the same argument in $\mathbb{R}^2$.*

### 3.6.2 Dyadic bilinear estimates

We are now looking to improve (3.6.8) in the case $M_{min} \geqslant 1$. We mainly follow [Zha15, Lemma 3.1]. However, in our situation the frequency for the $x$ variable lives in $\mathbb{R}$ and not in $\mathbb{Z}$, and thus the worst case of [Zha15, Lemma 3.1] (when $K_{med} \lesssim M_{max}M_{min}$) is avoided. So, using that this frequency is allowed to vary in very small intervals, we are able to recover the same result as in [IKT08, Lemma 5.1(a)]. Again, we will crucially use lemmas 3.6.2 and 3.6.3.

**Proposition 3.6.7**

*Let $M_i, K_i \in 2^{\mathbb{N}}$ and $f_i : \mathbb{R}^2 \times \lambda^{-1}\mathbb{Z} \to \mathbb{R}^+$, $i = 1, 2, 3$, be such that $f_i \in L^2(\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z})$ with $\operatorname{supp} f_i \subset D_{\lambda, M_i, \leqslant K_i}$.*
*If $K_{max} \leqslant 10^{-10} M_1 M_2 M_3$ and $K_{med} \gtrsim M_{max}$, then*

$$\int_{\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}} f_1 \star f_2 \cdot f_3 \mathrm{d}\tau \mathrm{d}\xi (\mathrm{d}q)_\lambda \lesssim \left( \frac{K_1 K_2 K_3}{M_1 M_2 M_3} \right)^{1/2} ||f_1||_{L^2} ||f_2||_{L^2} ||f_3||_{L^2} \tag{3.6.10}$$

**Proof :**
We begin as in [IKT08, Lemma 5.1(a)]. Defining

$$\mathcal{I}(f_1, f_2, f_3) := \int_{\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}} f_1 \star f_2 \cdot f_3 \mathrm{d}\tau \mathrm{d}\xi (\mathrm{d}q)_\lambda$$

we observe that

$$\mathcal{I}(f_1, f_2, f_3) = \mathcal{I}(\widetilde{f_1}, f_3, f_2) = \mathcal{I}(\widetilde{f_2}, f_3, f_1) \tag{3.6.11}$$

where we define $\widetilde{f}(x) := f(-x)$. Thus, as $\left|\left|\widetilde{f}\right|\right|_{L^2} = ||f||_{L^2}$, up to replacing $f_i$ by $\widetilde{f_i}$, we may assume $K_1 \leqslant K_2 \leqslant K_3$.
Moreover, since the expression is symmetrical in $f_1, f_2$ we can assume $M_2 \leqslant M_1$.
We first write

$$\begin{aligned}\mathcal{I}(f_1, f_2, f_3) &= \int_{\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}} f_1 \star f_2 \cdot f_3 \mathrm{d}\tau \mathrm{d}\xi (\mathrm{d}q)_\lambda \\ &= \int_{\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}} \int_{\mathbb{R} \times \mathbb{R}^+ \times \lambda^{-1}\mathbb{Z}} f_1(\tau_1, \zeta_1) f_2(\tau_2, \zeta_2) f_3(\tau_1 + \tau_2, \zeta_1 + \zeta_2) \mathrm{d}\tau_1 \mathrm{d}\tau_2 \mathrm{d}\zeta_1 \mathrm{d}\zeta_2\end{aligned}$$

Defining $f_i^{\#}(\theta, \zeta) := f_i(\theta + \omega(\zeta), \zeta)$ we get $\left|\left|f_i^{\#}\right|\right|_{L^2} = ||f_i||_{L^2}$ and $\operatorname{supp} f_i^{\#} \subset \{|\theta| \lesssim K_i, |\xi| \sim M_i\}$. Changing variables, we have

$$\begin{aligned}\mathcal{I}(f_1, f_2, f_3) = \int_{\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}} \int_{\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z}} & f_1^{\#}(\theta_1, \zeta_1) f_2^{\#}(\theta_2, \zeta_2) \\ & \cdot f_3^{\#}(\theta_1 + \theta_2 + \Omega(\zeta_1, \zeta_2, -\zeta_1 - \zeta_2), \zeta_1 + \zeta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\zeta_1 \mathrm{d}\zeta_2\end{aligned}$$

where the resonant function

$$\Omega(\zeta_1, \zeta_2, -\zeta_1 - \zeta_2) = -\frac{\xi_1 \xi_2}{\xi_1 + \xi_2} \left\{ \sqrt{3}|\xi_1 + \xi_2| + \left| \frac{q_1}{\xi_1} - \frac{q_2}{\xi_2} \right| \right\} \left\{ \sqrt{3}|\xi_1 + \xi_2| - \left| \frac{q_1}{\xi_1} - \frac{q_2}{\xi_2} \right| \right\}$$

has been defined in (3.6.9) in the proof of the previous proposition.

Thus

$$\mathcal{I}(f_1,f_2,f_3)=\int_A f_1^{\#}(\theta_1,\zeta_1)f_2^{\#}(\theta_2,\zeta_2)$$
$$\cdot f_3^{\#}(\theta_1+\theta_2+\Omega(\zeta_1,\zeta_2,-\zeta_1-\zeta_2),\zeta_1+\zeta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\zeta_1\mathrm{d}\zeta_2$$

with

$$A:=\big\{(\theta_1,\zeta_1,\theta_2,\zeta_2)\in(\mathbb{R}^2\times\lambda^{-1}\mathbb{Z})^2,\ |\xi_i|\sim M_i,\ |\xi_1+\xi_2|\sim M_3,\ |\theta_i|\lesssim K_i,$$
$$|\theta_1+\theta_2+\Omega(\zeta_1,\zeta_2,-\zeta_1-\zeta_2)|\lesssim K_3,\quad i=1,2\big\}$$

We can decompose $A\subset I_{\leqslant K_1}\times I_{\leqslant K_2}\times B$ with $B$ defined as

$$B:=\big\{(\zeta_1,\zeta_2)\in(\mathbb{R}\times\lambda^{-1}\mathbb{Z})^2,\ |\xi_i|\sim M_i, i=1,2,\ |\xi_1+\xi_2|\sim M_3,$$
$$|\Omega|\lesssim K_3\big\}\quad(3.6.12)$$

We can further split

$$B=\bigsqcup_{|\ell|\lesssim K_3/K_2}B_\ell$$

with

$$B_\ell:=\{(\zeta_1,\zeta_2)\in B,\ \Omega\in[\ell K_2;(\ell+1)K_2]\}\quad(3.6.13)$$

and as well for $f_3$ :

$$f_3^{\#}=\sum_{|\ell|\lesssim K_3/K_2}f_{3,\ell}^{\#}\text{ with }f_{3,\ell}^{\#}(\theta,\xi,q):=\mathbb{1}_{[\ell K_2,(\ell+1)K_2]}(\theta)f_3^{\#}(\theta,\xi,q)\quad(3.6.14)$$

Next, using Cauchy-Schwarz inequality in $\theta_2$ then $\theta_1$, we obtain

$$\mathcal{I}(f_1,f_2,f_3)\leqslant\sum_{|\ell|\lesssim K_3/K_2}\int_{I_{\leqslant K_1}\times B_\ell}|f_1^{\#}(\theta_1,\xi_1,q_1)|\left|\left|f_2^{\#}(\theta_2,\xi_2,q_2)\right|\right|_{L^2_{\theta_2}}$$
$$\cdot\left|\left|f_{3,\ell}^{\#}(\theta_1+\theta_2+\Omega,\xi_1+\xi_2,q_1+q_2)\right|\right|_{L^2_{\theta_2}}\mathrm{d}\theta_1\mathrm{d}\xi_1\mathrm{d}\xi_2(\mathrm{d}q_1)_\lambda(\mathrm{d}q_2)_\lambda$$
$$\lesssim K_1^{1/2}\sum_{|\ell|\lesssim K_3/K_2}\int_{B_\ell}\left|\left|f_1^{\#}(\theta_1,\xi_1,q_1)\right|\right|_{L^2_{\theta_1}}\left|\left|f_2^{\#}(\theta_2,\xi_2,q_2)\right|\right|_{L^2_{\theta_2}}$$
$$\cdot\left|\left|f_{3,\ell}^{\#}(\theta,\xi_1+\xi_2,q_1+q_2)\right|\right|_{L^2_\theta}\mathrm{d}\xi_1\mathrm{d}\xi_2(\mathrm{d}q_1)_\lambda(\mathrm{d}q_2)_\lambda$$

This allows us to work with functions depending on $(\xi_1,q_1),(\xi_2,q_2)$ only, losing just a factor $K_1^{1/2}$ in the process. The informations $|\Omega|\lesssim K_3$ and $\mathrm{supp}f_3\subset I_{\leqslant K_3}\times\mathfrak{I}_{M_3}\times\lambda^{-1}\mathbb{Z}$ have been kept in the decomposition on $\ell$ of $B$ and $f_3^{\#}$.

Finally, defining

$$g_i(\xi_i,q_i):=\left|\left|f_i^{\#}(\theta_i,\xi_i,q_i)\right|\right|_{L^2_{\theta_i}},\ i=1,2\text{ and }g_{3,\ell}(\xi,q):=\left|\left|f_{3,\ell}^{\#}(\theta,\xi,q)\right|\right|_{L^2_\theta}$$

and writing

$$J_\ell(g_1,g_2,g_{3,\ell}):=\int_{B_\ell}g_1(\xi_1,q_1)g_2(\xi_2,q_2)g_{3,\ell}(\xi_1+\xi_2,q_1+q_2)\mathrm{d}\xi_1\mathrm{d}\xi_2(\mathrm{d}q_1)_\lambda(\mathrm{d}q_2)_\lambda\quad(3.6.15)$$

it suffices to prove that

$$J := \sum_{\ell} J_\ell(g_1, g_2, g_{3,\ell}) \lesssim \left(\frac{K_2 K_3}{M_1 M_2 M_3}\right)^{1/2} ||g_1||_{L^2_{\xi_1,q_1}} ||g_2||_{L^2_{\xi_2,q_2}} \left\{\sum_{\ell} ||g_{3,\ell}||^2_{L^2_{\xi,q}}\right\}^{1/2} \tag{3.6.16}$$

As we are in the regime $K_{max} \lesssim M_1 M_2 M_3$, $\Omega$ is close to zero. Since $q_i \in \lambda^{-1}\mathbb{Z}$, we cannot just make a change of variables as in [IKT08, Lemma 5.1(a)]. Thus, to take into account that $(\sqrt{3}\xi_1 + \sqrt{3}\xi_2)^2 \sim \left(\frac{q_1}{\xi_1} - \frac{q_2}{\xi_2}\right)^2$, we split $B_\ell$ depending on the values of $q_1$ and $q_2$.

First, as in [IKT08, Lemma 5.1(a)], we can split

$$B_\ell := B_\ell^{++} \sqcup B_\ell^{+-} \sqcup B_\ell^{-+} \sqcup B_\ell^{--}$$

with

$$B_\ell^{\varepsilon_1,\varepsilon_2} := \left\{(\xi_1, q_1), (\xi_2, q_2) \in B_\ell,\ \text{sign}(\xi_1 + \xi_2) = \varepsilon_1,\ \text{sign}\left(\frac{q_1}{\xi_1} - \frac{q_2}{\xi_2}\right) = \varepsilon_2\right\}$$

where $\varepsilon_i \in \{\pm 1\}$.

Since the transformations $(\xi_1, q_1), (\xi_2, q_2) \mapsto (\varepsilon_1\xi_1, \varepsilon_2 q_2), (\varepsilon_1\xi_1, \varepsilon_2 q_2)$ maps $B_\ell^{\varepsilon_1,\varepsilon_2}$ to $B_\ell^{++}$, it suffices to estimate

$$J_\ell^{++}(g_1, g_2, g_{3,\ell}) := \int_{B_\ell^{++}} g_1(\xi_1, q_1) g_2(\xi_2, q_2) g_{3,\ell}(\xi_1 + \xi_2, q_1 + q_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2 (\mathrm{d}q_1)_\lambda (\mathrm{d}q_2)_\lambda$$

Moreover, the definition of $\Omega$ and the condition $|\Omega| \lesssim K_3$ give

$$\left|\sqrt{3}(\xi_1 + \xi_2) - \left(\frac{q_1}{\xi_1} - \frac{q_2}{\xi_2}\right)\right| \leqslant \frac{|\Omega|}{|\xi_1 \xi_2|} \lesssim \frac{K_3}{M_1 M_2} \tag{3.6.17}$$

on $B_\ell^{++}$.

Now, we can define

$$\mathcal{Q}_1(\xi_1, q_1, \xi_2, q_2) := \left\lfloor \frac{M_1 M_2}{K_2}(q_1 - \sqrt{3}\xi_1^2)/\xi_1 \right\rfloor \in \mathbb{Z} \tag{3.6.18}$$

and

$$\mathcal{Q}_2(\xi_1, q_1, \xi_2, q_2) := \mathcal{Q}_1(\xi_1, q_1, \xi_2, q_2) - \left\lfloor \frac{M_1 M_2}{K_2}(q_2 + \sqrt{3}\xi_2^2)/\xi_2 \right\rfloor \in \mathbb{Z} \tag{3.6.19}$$

So we can split $B_\ell^{++}$ according to the level sets of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ :

$$B_\ell^{++} = \bigsqcup_{Q_1, Q_2 \in \mathbb{Z}} B_{\ell,Q_1,Q_2}$$

where $B_{\ell,Q_1,Q_2}$ is defined as

$$B_{\ell,Q_1,Q_2} := \left\{(\xi_1, q_1), (\xi_2, q_2) \in B_\ell^{++},\ \mathcal{Q}_1(\xi_1, q_1, \xi_2, q_2) = Q_1,\ \mathcal{Q}_2(\xi_1, q_1, \xi_2, q_2) = Q_2\right\}$$

From definitions (3.6.18) and (3.6.19), for $(\xi_1, q_1), (\xi_2, q_2) \in B_{\ell,Q_1,Q_2}$, $Q_2$ is such that

$$Q_2 = \left\lfloor \frac{M_1 M_2}{K_2}\left(\frac{q_1}{\xi_1} - \frac{q_2}{\xi_2} - \sqrt{3}(\xi_1 + \xi_2)\right)\right\rfloor$$
$$\text{or } Q_2 = \left\lfloor \frac{M_1 M_2}{K_2}\left(\frac{q_1}{\xi_1} - \frac{q_2}{\xi_2} - \sqrt{3}(\xi_1 + \xi_2)\right)\right\rfloor + 1$$

Thus,

$$\frac{q_1}{\xi_1} - \frac{q_2}{\xi_2} - \sqrt{3}(\xi_1 + \xi_2) \in \left[ \frac{K_2}{M_1 M_2}(Q_2 - 1)\ ;\ \frac{K_2}{M_1 M_2}(Q_2 + 1) \right] \tag{3.6.20}$$

Finally, if $(\xi_1, q_1), (\xi_2, q_2) \in B_{\ell, Q_1, Q_2}$ we obtain from (3.6.9) and (3.6.20) that

$$\begin{aligned}\Omega(\xi_1, q_1, \xi_2, q_2) &= \frac{\xi_1 \xi_2}{\xi_1 + \xi_2} \frac{K_2}{M_1 M_2} (Q_2 + \nu) \left( \left| \frac{q_1}{\xi_1} - \frac{q_2}{\xi_2} \right| + \sqrt{3}\, |\xi_1 + \xi_2| \right) \\ &= \frac{\xi_1 \xi_2 K_2}{M_1 M_2} (Q_2 + \nu) \left( 2\sqrt{3} + \frac{K_2}{M_1 M_2} \frac{Q_2 + \nu}{(\xi_1 + \xi_2)} \right) \end{aligned} \tag{3.6.21}$$

with

$$|\nu| \leqslant 1$$

The choice of the parameter $\frac{K_2}{M_1 M_2}$ in the definitions of $Q_i$ allows us to have $\frac{q_1}{\xi_1}$ and $\frac{q_2}{\xi_2}$ of the same order, and thus to keep an error $\nu$ of size $O(1)$ in this "change of variables". The measure of the $q_i$-sections of $B_{\ell, Q_1, Q_2}$ is then controlled with $\frac{K_2 M_i}{M_1 M_2} \gtrsim 1$ (as $K_2 \gtrsim M_{max}$), $i = 1, 2$.
Using (3.6.17), we get

$$|Q_2| \lesssim \frac{K_3}{K_2}$$

Moreover, by definition

$$\forall (\xi_1, q_1), (\xi_2, q_2) \in B_\ell,\ \ell = \left\lfloor \frac{\Omega(\xi_1, q_1, \xi_2, q_2)}{K_2} \right\rfloor$$

and so a key remark is that if $(\xi_1, q_1), (\xi_2, q_2) \in B_{\ell, Q_1, Q_2}$ :

$$\ell = \ell(\xi_1, \xi_2, Q_2) = \left\lfloor \frac{\xi_1 \xi_2}{M_1 M_2} (Q_2 + \nu) \left( 2\sqrt{3} + \frac{K_2}{M_1 M_2} \frac{Q_2 + \nu}{\xi_1 + \xi_2} \right) \right\rfloor \tag{3.6.22}$$

Using that $|\xi_i| \sim M_i$, $\xi_1 + \xi_2 \sim M_3$, $|Q_2| \lesssim K_3/K_2$ and that we assumed $K_3 \leqslant 10^{-10} M_1 M_2 M_3$, we get that

$$\left| \frac{K_2}{M_1 M_2} \frac{Q_2 + \nu}{\xi_1 + \xi_2} \right| \leqslant 10^{-5}$$

which means that for any fixed $Q_1, Q_2$ there is at most 10 possible values for $\ell$ such that $B_{\ell, Q_1, Q_2}$ is non empty.

Let us write $J_{\ell, Q_1, Q_2}$ the contribution of the region $B_{\ell, Q_1, Q_2}$ in the integral $J_\ell^{++}$. To control $J_{\ell, Q_1, Q_2}$ we first use Cauchy-Schwarz inequality in $q_1, q_2, \xi_1, \xi_2$ :

$$\begin{aligned} J_{\ell, Q_1, Q_2} \lesssim\ & ||g_1||_{L^2(B^1_{Q_1})}\, ||g_2||_{L^2(B^2_{Q_1, Q_2})} \\ & \cdot \left\{ \int_{B_{\ell, Q_1, Q_2}} g^2_{3,\ell}(\xi_1 + \xi_2, q_1 + q_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2 (\mathrm{d}q_1)_\lambda (\mathrm{d}q_2)_\lambda \right\}^{1/2} \end{aligned}$$

where we define

$$\begin{aligned} B^1_{Q_1} := \big\{ & (\xi_1, q_1) \in \mathfrak{I}_{M_1} \times \lambda^{-1}\mathbb{Z}, \\ & \sqrt{3}\xi_1^2 + Q_1 \frac{K_2}{M_1 M_2} \xi_1 \leqslant q_1 < \sqrt{3}\xi_1^2 + (Q_1 + 1) \frac{K_2}{M_1 M_2} \xi_1 \Big\} \end{aligned} \tag{3.6.23}$$

and

$$B^2_{Q_1,Q_2} := \Big\{(\xi_2, q_2) \in \mathfrak{I}_{M_2} \times \lambda^{-1}\mathbb{Z}, \\ -\sqrt{3}\xi_2^2 + (Q_1 - Q_2)\frac{K_2}{M_1M_2}\xi_2 \leqslant q_2 < -\sqrt{3}\xi_2^2 + (Q_1 - Q_2 + 1)\frac{K_2}{M_1M_2}\xi_2\Big\} \quad (3.6.24)$$

Let us start by treating the integral over $B_{\ell,Q_1,Q_2}$.
If $(\xi_1, q_1), (\xi_2, q_2) \in B_{\ell,Q_1,Q_2}$, we can parametrize the $q_i$-sections with

$$r_1 := q_1 - \left[\sqrt{3}\xi_1^2 + Q_1\frac{K_2}{M_1M_2}\xi_1\right]_\lambda \in \lambda^{-1}\mathbb{Z}$$

and

$$r_2 := q_2 - \left[-\sqrt{3}\xi_2^2 + (Q_1 - Q_2)\frac{K_2}{M_1M_2}\xi_2\right]_\lambda \in \lambda^{-1}\mathbb{Z}$$

such that $0 \leqslant r_i \lesssim \dfrac{K_2M_i}{M_1M_2}$.
As we assumed $M_2 \leqslant M_1$, the $q_2$-sections of $B_{\ell,Q_1,Q_2}$ are then smaller than the $q_1$-sections, and thus $0 \leqslant r_1 + r_2 \lesssim r_1$. So if $\xi_1, \xi_2$ are fixed, we obtain :

$$\int\int \mathbb{1}_{B_{\ell,Q_1,Q_2}}(\xi_1, q_1, \xi_2, q_2) g_{3,\ell}^2(\xi_1 + \xi_2, q_1 + q_2)(\mathrm{d}q_1)_\lambda(\mathrm{d}q_2)_\lambda \\ = \int\int \mathbb{1}_{[0;K_2/M_2]}(r_1)\mathbb{1}_{[0;K_2/M_1]}(r_2) g_{3,\ell}^2\Big(\xi_1 + \xi_2, \\ \left[\sqrt{3}\xi_1^2 + Q_1\frac{K_2}{M_1M_2}\xi_1\right]_\lambda + r_1 + \left[-\sqrt{3}\xi_2^2 + (Q_1 - Q_2)\frac{K_2}{M_1M_2}\xi_2\right]_\lambda + r_2\Big)(\mathrm{d}r_1)_\lambda(\mathrm{d}r_2)_\lambda \\ \lesssim \frac{K_2}{M_1}\int \mathbb{1}_{[0;K_2/M_2]}(|r|) g_{3,\ell}^2\Big(\xi_1 + \xi_2, \\ \left[\sqrt{3}\xi_1^2 + Q_1\frac{K_2}{M_1M_2}\xi_1 - \sqrt{3}\xi_2^2 + (Q_1 - Q_2)\frac{K_2}{M_1M_2}\xi_2\right]_\lambda + r\Big)(\mathrm{d}r)_\lambda$$

The integral over $B_{\ell,Q_1,Q_2}$ is thus controlled by

$$J_{\ell,Q_1,Q_2} \lesssim \left(\frac{K_2}{M_1}\right)^{1/2} ||g_1||_{L^2(B^1_{Q_1})} ||g_2||_{L^2(B^2_{Q_1,Q_2})} \Big\{\int_{\mathbb{R}^2}\int \mathbb{1}_{[0;K_2/M_2}(|r|) g_{3,\ell}^2\Big(\xi_1 + \xi_2, \\ \left[\sqrt{3}\xi_1^2 + Q_1\frac{K_2}{M_1M_2}\xi_1 - \sqrt{3}\xi_2^2 + (Q_1 - Q_2)\frac{K_2}{M_1M_2}\xi_2\right]_\lambda + r\Big)\mathrm{d}\xi_1\mathrm{d}\xi_2(\mathrm{d}r)_\lambda\Big\}^{1/2}$$

It remains to sum those contributions : using the previous estimate and that for fixed $Q_1, Q_2$

the sum in $\ell$ runs over at most 10 integers,

$$\begin{aligned}J = &\sum_{|\ell|\lesssim K_3/K_2} \sum_{Q_1\in\mathbb{Z}} \sum_{|Q_2|\lesssim K_3/K_2} J_{\ell,Q_1,Q_2} \\ &\lesssim \sum_{Q_1\in\mathbb{Z}} \sum_{|Q_2|\lesssim K_3/K_2} \left(\frac{K_2}{M_1}\right)^{1/2} ||g_1||_{L^2(B^1_{Q_1})} ||g_2||_{L^2(B^2_{Q_1,Q_2})} \\ &\cdot \left\{ \sum_{|\ell|\lesssim K_3/K_2} \int_{\mathbb{R}^2}\int \mathbb{1}_{[0;K_2/M_2]}(|r|) g^2_{3,\ell}\Big(\xi_1+\xi_2, \right. \\ &\left. \left[\sqrt{3}\xi_1^2 + Q_1\frac{K_2}{M_1M_2}\xi_1 - \sqrt{3}\xi_2^2 + (Q_1-Q_2)\frac{K_2}{M_1M_2}\xi_2\right]_\lambda + r\Big)\, \mathrm{d}\xi_1\mathrm{d}\xi_2(\mathrm{d}r)_\lambda \right\}^{1/2}\end{aligned}$$

Next, a use of Cauchy-Schwarz inequality in $Q_2$ then $Q_1$ gives

$$\begin{aligned}J \lesssim &\left(\frac{K_2}{M_1}\right)^{1/2} \left(\sum_{Q_1\in\mathbb{Z}} ||g_1||^2_{L^2(B^1_{Q_1})}\right)^{1/2} \left(\sum_{Q_1\in\mathbb{Z}} \sum_{|Q_2|\lesssim K_3/K_2} ||g_2||^2_{L^2(B^2_{Q_1,Q_2})}\right)^{1/2} \\ &\cdot \left\{ \sup_{Q_1} \sum_{|Q_2|\lesssim K_3/K_2} \sum_{|\ell|\lesssim K_3/K_2} \int_{\mathbb{R}^2}\int \mathbb{1}_{[0;K_2/M_2]}(|r|) g^2_{3,\ell}\Big(\xi_1+\xi_2, \right. \\ &\left. \left[\sqrt{3}\xi_1^2 + Q_1\frac{K_2}{M_1M_2}\xi_1 - \sqrt{3}\xi_2^2 + (Q_1-Q_2)\frac{K_2}{M_1M_2}\xi_2\right]_\lambda + r\Big)\, \mathrm{d}\xi_1\mathrm{d}\xi_2(\mathrm{d}r)_\lambda \right\}^{1/2}\end{aligned}$$

Now, from the definitions of $B^1_{Q_1}$ (3.6.23) and $B^2_{Q_1,Q_2}$ (3.6.24) :

$$\left(\sum_{Q_1\in\mathbb{Z}} ||g_1||^2_{L^2(B^1_{Q_1})}\right)^{1/2} = ||g_1||_{L^2_{\xi_1,q_1}} = ||f_1||_{L^2}$$

and

$$\begin{aligned}\left(\sum_{Q_1\in\mathbb{Z}} \sum_{|Q_2|\lesssim K_3/K_2} ||g_2||^2_{L^2(B^2_{Q_1,Q_2})}\right)^{1/2} &\lesssim \left(\frac{K_3}{K_2}\right)^{1/2} \left(\sup_{Q_2} \sum_{Q_1\in\mathbb{Z}} ||g_2||^2_{L^2(B_{Q_1,Q_2})}\right)^{1/2} \\ &= \left(\frac{K_3}{K_2}\right)^{1/2} ||g_2||_{L^2_{\xi_2,q_2}} = \left(\frac{K_3}{K_2}\right)^{1/2} ||f_2||_{L^2}\end{aligned}$$

To conclude, it suffices to prove

$$\begin{aligned}&\sup_{Q_1} \sum_{|Q_2|\lesssim K_3/K_2} \sum_{|\ell|\lesssim K_3/K_2} \int_{\mathbb{R}^2}\int \mathbb{1}_{[0;K_2/M_2]}(|r|) g^2_{3,\ell}\Big(\xi_1+\xi_2, \\ &\left[\sqrt{3}\xi_1^2 + Q_1\frac{K_2}{M_1M_2}\xi_1 - \sqrt{3}\xi_2^2 + (Q_1-Q_2)\frac{K_2}{M_1M_2}\xi_2\right]_\lambda + r\Big)\, \mathrm{d}\xi_1\mathrm{d}\xi_2(\mathrm{d}r)_\lambda \\ &\lesssim \frac{K_2}{M_2M_3} ||f_3||_{L^2} \quad (3.6.25)\end{aligned}$$

Here, we can see the interest of splitting $f_3^{\#}$ over $\ell$ : the sum over $\ell$ is controlled by the sum over $Q_2$ thanks to (3.6.22), whereas a direct estimate on this sum would lose an additional factor $K_3/K_2$ (or in other words, when $\xi_1, \xi_2, Q_2$ are fixed, we do not have the contribution of the full $L^2$ norm of $f_3^{\#}$ in the $\theta$ variable, which allows us to sum those contributions without losing an additionnal factor).

We begin the proof of (3.6.25) with the change of variables $\xi_1 \mapsto \xi := \xi_1 + \xi_2$ : the left-hand side now reads

$$\sup_{Q_1} \sum_{|Q_2| \lesssim K_3/K_2} \sum_{|\ell| \lesssim K_3/K_2} \int_{\mathbb{R}^2} \int \mathbb{1}_{[0;K_2/M_2]}(|r|) g_{3,\ell}^2 \Bigg(\xi, \left[\sqrt{3}\xi(\xi - 2\xi_2) + Q_1 \frac{K_2}{M_1 M_2}\xi - Q_2 \frac{K_2}{M_1 M_2}\xi_2\right]_\lambda + r\Bigg) \mathrm{d}\xi_2 \mathrm{d}\xi (\mathrm{d}r)_\lambda$$

Now, using (3.6.22) and the definition of $g_{3,\ell}$, we have that for fixed $\xi, Q_1, \xi_2, Q_2, r$ :

$$\begin{aligned} &\sum_{|\ell| \lesssim K_3/K_2} g_{3,\ell}^2 \left(\xi, \left[\sqrt{3}\xi(\xi - 2\xi_2) + Q_1 \frac{K_2}{M_1 M_2}\xi - Q_2 \frac{K_2}{M_1 M_2}\xi_2\right]_\lambda + r\right) \\ &\quad \lesssim \int_{\mathbb{R}} \mathbb{1}\left(\theta \in \left[\frac{(\xi - \xi_2)\xi_2 K_2}{M_1 M_2}(Q_2 - 2)\left(2\sqrt{3} + \frac{K_2}{M_1 M_2}\frac{Q_2 - 2}{\xi}\right) ; \right.\right. \\ &\qquad\qquad \left.\left. \frac{(\xi - \xi_2)\xi_2 K_2}{M_1 M_2}(Q_2 + 2)\left(2\sqrt{3} + \frac{K_2}{M_1 M_2}\frac{Q_2 + 2}{\xi}\right)\right]\right) \\ &\qquad\qquad \cdot (f_3^{\#})^2 \left(\theta, \xi, \left[\sqrt{3}\xi(\xi - 2\xi_2) + Q_1 \frac{K_2}{M_1 M_2}\xi - Q_2 \frac{K_2}{M_1 M_2}\xi_2\right]_\lambda + r\right) \mathrm{d}\theta \end{aligned}$$

Now, fixing only $\xi$, and $Q_1$, integrating in $\xi_2$ and $r$ and summing over $Q_2$, we can write the previous term as

$$\sum_{|Q_2| \lesssim K_3/K_2} \int_{\mathfrak{I}_{M_2}} \int \mathbb{1}_{[0;K_2/M_2]}(|r|) \int_{\mathbb{R}} \mathbb{1}\{\theta \in I(\xi, \xi_2, Q_2)\} \cdot (f_3^{\#})^2 (\theta, \xi, [\varphi(\xi, Q_1, \xi_2, Q_2)]_\lambda + r) \mathrm{d}\theta (\mathrm{d}r)_\lambda \mathrm{d}\xi_2$$

where the interval $I(\xi, \xi_2, Q_2)$ is defined as

$$I(\xi, \xi_2, Q_2) := \left[\frac{(\xi - \xi_2)\xi_2 K_2}{M_1 M_2}(Q_2 - 2)\left(2\sqrt{3} + \frac{K_2}{M_1 M_2}\frac{Q_2 - 2}{\xi}\right) ; \frac{(\xi - \xi_2)\xi_2 K_2}{M_1 M_2}(Q_2 + 2)\left(2\sqrt{3} + \frac{K_2}{M_1 M_2}\frac{Q_2 + 2}{\xi}\right)\right]$$

and the function $\varphi$ is defined as

$$\varphi(\xi, Q_1, \xi_2, Q_2) := \sqrt{3}\xi(\xi - 2\xi_2) + Q_1 \frac{K_2}{M_1 M_2}\xi - Q_2 \frac{K_2}{M_1 M_2}\xi_2$$

In order to recover the $L^2$ norm of $f_3^{\#}$ in $q$, we decompose the previous term in

$$\lambda \int_{\lambda^{-1}\mathbb{Z}} \sum_{|Q_2| \lesssim K_3/K_2} \int \int_{\Lambda_n(\xi, Q_1, Q_2)} \int_{\mathbb{R}} \mathbb{1}\{\theta \in I(\xi, \xi_2, Q_2)\} \cdot (f_3^{\#})^2 (\theta, \xi, n) \mathrm{d}\theta (\mathrm{d}\xi_2 (\mathrm{d}r)_\lambda)(\mathrm{d}n)_\lambda$$

where the set $\Lambda_n(\xi, Q_1, Q_2) \subset \mathbb{R} \times \lambda^{-1}\mathbb{Z}$ for $n \in \lambda^{-1}\mathbb{Z}$ is defined as

$$\Lambda_n(\xi, Q_1, Q_2) := \Big\{ (\xi_2, r) \in \mathfrak{I}_{M_2} \times [-\frac{K_2}{M_2}; \frac{K_2}{M_2}], \qquad \varphi(\xi, Q_1, \xi_2, Q_2) \in [n - r; n + \lambda^{-1} - r[ \Big\}$$

First, using the localizations $|\xi| \sim M_3$, $|\xi_2| \sim M_2$ and $|\xi - \xi_2| \sim M_1$ and the conditions $|Q_2| \lesssim K_3/K_2$ and $K_3 \leqslant 10^{-10} M_1 M_2 M_3$, we have for any $\xi, \xi_2, Q_2$ :

$$I(\xi, \xi_2, Q_2) \subset \big\{ |\theta| \in \big[ c^{-1} K_2(Q_2 - 2), cK_2(Q_2 + 2) \big] \big\}$$

for an absolute constant $c > 0$.

Thus we are left with estimating

$$\lambda \int_{\lambda^{-1}\mathbb{Z}} \sum_{|Q_2| \lesssim K_3/K_2} \int_{\mathbb{R}} \mathbb{1} \left\{ |\theta| \in [c^{-1} K_2(Q_2 - 2); cK_2(Q_2 + 2)] \right\} \cdot |\Lambda_n(\xi, Q_1, Q_2)| \, (f_3^{\#})^2(\theta, \xi, n) \mathrm{d}\theta (\mathrm{d}n)_\lambda \quad (3.6.26)$$

We trivially control the measure of the $r$-sections of $\Lambda_n$ with $2\dfrac{K_2}{M_2}$. It remains to estimate the measure of the projection of $\Lambda_n$ on the $\xi_2$ axis, uniformly in $n, \xi, Q_1$ and $Q_2$. To do so, we are going to make a good use of lemma 3.6.3. We are then left to compute $\dfrac{\partial \varphi}{\partial \xi_2}$ :

$$\frac{\partial \varphi}{\partial \xi_2} = -2\sqrt{3}\xi - Q_2 \frac{K_2}{M_1 M_2}$$

Now, as $|Q_2| \lesssim K_3/K_2$ and $K_3 \leqslant 10^{-10} M_1 M_2 M_3$, we obtain that

$$\left| \frac{\partial \varphi}{\partial \xi_2} \right| \sim 2\sqrt{3}|\xi| \sim M_3$$

So, applying (3.6.3), we get that the projection of $\Lambda_n(\xi, Q_1, Q_2)$ on the $\xi_2$ axis is controlled by $\lambda^{-1} M_3^{-1}$. A use of lemma 3.6.2 finally leads to

$$|\Lambda_n(\xi, Q_1, Q_2)| \lesssim \lambda^{-1} \frac{K_2}{M_2 M_3}$$

uniformly in $n, \xi, Q_1, Q_2$.

Getting back to (3.6.26), we have

$$\begin{aligned}(3.6.26) &\lesssim \frac{K_2}{M_2 M_3} \int_{\lambda^{-1}\mathbb{Z}} \int_{\mathbb{R}} \left( \sum_{|Q_2| \lesssim K_3/K_2} \mathbb{1} \left\{ \theta \in [c^{-1} K_2(Q_2 - 2); cK_2(Q_2 + 2)] \right\} \right) \cdot (f_3^{\#})^2(\theta, \xi, n) \mathrm{d}\theta (\mathrm{d}n)_\lambda \\ &\lesssim \frac{K_2}{M_2 M_3} \int_{\lambda^{-1}\mathbb{Z}} \int_{\mathbb{R}} \mathbb{1} \left\{ |\theta| \in I_{\leqslant K_3} \right\} (f_3^{\#})^2(\theta, \xi, n) \mathrm{d}\theta (\mathrm{d}n)_\lambda\end{aligned}$$

Now, neglecting the $\theta$ localization and integrating in $\xi$, we finally get (3.6.25), which completes the proof of the proposition.

□

**Remark 3.6.8.** *In the case $(x,y) \in \mathbb{T}^2$ ([Zha15, Lemma 3.1]), we can still use lemma 3.6.3, but since $\xi_2 \in \mathbb{Z}$ in that case, we have to use (3.6.4) instead of (3.6.3), and thus we have the rougher estimate*

$$|\Lambda_n| \lesssim \frac{K_2}{M_2}\left(1 + M_3^{-1}\right) \lesssim \frac{K_2}{M_2}$$

*as $M_i \geqslant 1$ for localized functions on $\mathbb{T}^2$. This is the main obstacle to recover the same estimate as in $\mathbb{R}^2$ or $\mathbb{R}\times\mathbb{T}$, and the cause of the logarithmic divergence in the energy estimate.*

The following corollary summarizes the estimates on $\displaystyle\int f_1 \star f_2 \cdot f_3$ according to the relations between the $M$'s and the $K$'s :

**Corollary 3.6.9**

*Let $f_i \in L^2(\mathbb{R}^2\times\lambda^{-1}\mathbb{Z})$ be positive functions with the support condition* $\mathrm{supp} f_i \subset D_{\lambda,M_i,\leqslant K_i}$, *$i = 1,2,3$. We assume $K_{med} \geqslant M_{max} \geqslant 1$.*

*(a) If $K_{max} \leqslant 10^{-10} M_1M_2M_3$ then*

$$\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_1 \star f_2 \cdot f_3 \lesssim \left(M_{min} \wedge M_{min}^{-1}\right)^{1/2} M_{max}^{-1} \prod_{i=1}^{3} K_i^{1/2} \left|\left|f_i\right|\right|_{L^2} \tag{3.6.27}$$

*(b) If $K_{max} \gtrsim M_1M_2M_3$ and $(M_i,K_i) = (M_{min},K_{max})$ for an $i \in \{1,2,3\}$ then*

$$\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_1 \star f_2 \cdot f_3 \lesssim (1 \wedge M_{min})^{1/4} M_{max}^{-1} \prod_{i=1}^{3} K_i^{1/2} \left|\left|f_i\right|\right|_{L^2} \tag{3.6.28}$$

*(c) If $K_{max} \gtrsim M_1M_2M_3$ but $(M_i,K_i) \neq (M_{min},K_{max})$ for any $i = 1,2,3$ then*

$$\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_1 \star f_2 \cdot f_3 \lesssim (1 \vee M_{min})^{1/4} M_{max}^{-5/4} \prod_{i=1}^{3} K_i^{1/2} \left|\left|f_i\right|\right|_{L^2} \tag{3.6.29}$$

**Proof :**
Using the symmetry property (3.6.11), we can assume $K_3 = K_{max}$. Note that, since $M_{max} \geqslant 1$ and in order for the integral to be non zero, we must have $(1 \vee M_{min}) \lesssim M_{med} \sim M_{max}$. Then we treat the different cases.
**Case (a) :** This has already been proven in the previous proposition in the case $M_{min} \geqslant 1$. If $M_{min} \leqslant 1$, (3.6.27) follows from (3.6.8), since $K_3 = K_{max} \geqslant (K_1 \vee K_2) \geqslant M_{max}$.
**Case (b) :** $M_3 = M_{min}$. Then, if $M_3 \geqslant 1$, (3.6.28) follows from (3.6.7) since

$$\left\langle (K_1 \vee K_2)^{1/4} (M_1 \wedge M_2)^{1/4} \right\rangle \lesssim (K_1 \vee K_2)^{1/2}$$

as $(K_1 \vee K_2) \gtrsim M_{max}$.
If $M_3 \leqslant 1$, since this is symmetrical in $f_1$ and $f_2$ we may assume that $K_1 = K_1 \wedge K_2$. Then we apply (3.6.7) with $f_1$ and $f_3$ to get (3.6.28) since $K_3^{-1/4} \lesssim M_{min}^{-1/4} M_{max}^{-1/2}$ and $K_2^{-1/2} = K_{med}^{-1/2} \lesssim M_{max}^{-1/2}$.
**Case (c) :** Again, (3.6.29) follows from (3.6.7) since

$$\left\langle (K_1 \vee K_2)^{1/4} (M_1 \wedge M_2)^{1/4} \right\rangle \lesssim (K_1 \vee K_2)^{1/2} M_{max}^{-1/4} (1 \vee M_{min})^{1/4}$$

□

We conclude this section by stating another estimate which takes into account the weight in the definition of the energy space :

**Proposition 3.6.10**

*Let $f_i \in L^2(\mathbb{R}^2 \times \lambda^{-1}\mathbb{Z})$ be positive functions with the support condition* $\mathrm{supp} f_i \subset D_{\lambda,M_i,K_i}$*, $i = 1, 2$ for $M_3 > 0$, $K_3 \geqslant 1$. Then*

$$\left|\left|\mathbb{1}_{D_{\lambda,M_3,K_3}} \cdot f_1 \star f_2\right|\right|_{L^2} \lesssim (1 \vee M_1) M_{min}^{1/2} K_{min}^{1/2} \left|\left|p \cdot f_1\right|\right|_{L^2} \left|\left|f_2\right|\right|_{L^2} \tag{3.6.30}$$

**Proof :**

We follow [IKT08, Corollary 5.3 (b)&(c)] : we split the cases $M_1 \lesssim 1$ or $M_1 \gtrsim 1$ and we decompose $f_1$ on its $y$ frequency in order to estimate $p(\xi, q) \sim 1 + \dfrac{|q|}{|\xi|\langle\xi\rangle}$.

**Case 1 :** If $M_1 \geqslant 1$.

We then have $p(\xi, q) \sim 1 + \dfrac{|q|}{|\xi|^2}$. We split

$$f_1 = \sum_{L \geqslant M_1^2} f_1^L = \mathbb{1}_{I_{\leqslant M_1^2}}(q) f_1 + \sum_{L > M_1^2} \mathbb{1}_{I_L}(q) f_1$$

such that

$$\left|\left|\mathbb{1}_{D_{\lambda,M_3,K_3}} \cdot f_1 \star f_2\right|\right|_{L^2} \lesssim \sum_{L \geqslant M_1^2} L^{1/2} M_{min}^{1/2} K_{min}^{1/2} \left|\left|f_1^L\right|\right|_{L^2} \left|\left|f_2\right|\right|_{L^2}$$

after using (3.6.1). Now, for $L = M_1^2$ we have $L^{-1/2} p \sim M_1^{-1}(1 + M_1^{-2}|q|) \gtrsim M_1^{-1} = L^{1/2} M_1^{-2}$, and for $L > M_1^2$ we also have $L^{-1/2} p \sim L^{-1/2}(1 + L M_1^{-2}) \gtrsim L^{1/2} M_1^{-2}$. Thus, using Cauchy-Schwarz inequality in $L$, we obtain

$$\begin{aligned}\left|\left|\mathbb{1}_{D_{\lambda,M_3,K_3}} \cdot f_1 \star f_2\right|\right|_{L^2} &\lesssim M_{min}^{1/2} K_{min}^{1/2} \left|\left|f_2\right|\right|_{L^2} \sum_{L \geqslant M_1^2} L^{-1/2} M_1^2 \left|\left|p \cdot f_1^L\right|\right|_{L^2} \\ &\lesssim M_1^2 M_{min}^{1/2} K_{min}^{1/2} \cdot M_1^{-1} \left|\left|p \cdot f_1\right|\right|_{L^2}\end{aligned}$$

**Case 2 :** If $M_1 \leqslant 1$.

This time, we split the $y$ frequency for $L \geqslant 1$ since for $M_1 < \lambda^{-1}$ there is just the frequency $q = 0$ :

$$f_1 = \sum_{L \geqslant 1} f_1^L = \mathbb{1}_{I_{\leqslant 1}}(q) f_1 + \sum_{L > 1} \mathbb{1}_{I_L}(q) f_1$$

For $L = 1$, we have $L^{-1/2} p \gtrsim 1 = L^{1/2}$, and for $L > 1$, we also have $L^{-1/2} p \gtrsim L^{1/2} M_1^{-1} \gtrsim L^{1/2}$. Thus, using again (3.6.1) and then Cauchy-Schwarz inequality in $L$, we only get in that case

$$\begin{aligned}\left|\left|\mathbb{1}_{D_{\lambda,M_3,K_3}} \cdot f_1 \star f_2\right|\right|_{L^2} &\lesssim \sum_{L \geqslant 1} L^{1/2} M_{min}^{1/2} K_{min}^{1/2} \left|\left|f_1^L\right|\right|_{L^2} \left|\left|f_2\right|\right|_{L^2} \\ &\lesssim M_{min}^{1/2} K_{min}^{1/2} \sum_{L \geqslant 1} L^{-1/2} \left|\left|p \cdot f_1^L\right|\right|_{L^2} \left|\left|f_2\right|\right|_{L^2} \\ &\lesssim M_{min}^{1/2} K_{min}^{1/2} \left|\left|p \cdot f_1\right|\right|_{L^2} \left|\left|f_2\right|\right|_{L^2}\end{aligned}$$

□

## 3.7 Bilinear estimates

The aim of this section is to prove (3.1.12) and (3.1.15). We will treat separately the interactions *Low* × *High* → *High*, *High* × *High* → *Low* and *Low* × *Low* → *Low*. Those are the only possible interactions, since for functions $f_i$ localized in $|\xi_i| \sim M_i$, we have

$$\int f_1 \star f_2 \cdot f_3 \neq 0 \Rightarrow M_{min} \lesssim M_{med} \sim M_{max}$$

### 3.7.1 For the equation

We first prove (3.1.12).

**Lemma 3.7.1 (*Low* × *High* → *High*)**

*Let $M_1, M_2, M_3 \in 2^{\mathbb{Z}}$ with $(1 \vee M_1) \lesssim M_2 \sim M_3$ and $b_1 \in [0; 1/2[$. Then for $u_{M_1} \in N^0_{\lambda,M_1}$ and $v_{M_2} \in N^0_{\lambda,M_2}$, we have*

$$||P_{M_3}\partial_x (u_{M_1} \cdot v_{M_2})||_{N^{b_1}_{\lambda,M_3}} \lesssim M_1^{1/2} ||u_{M_1}||_{F^0_{\lambda,M_1}} ||v_{M_2}||_{F^0_{\lambda,M_2}} \tag{3.7.1}$$

**Proof :**
By definition, the left-hand side of (3.7.1) is

$$\sup_{t_{M_3}\in\mathbb{R}} \left|\left|(\tau - \omega + iM_3)^{-1} p \cdot \mathcal{F}\left\{\chi_{M_3^{-1}}(t - t_{M_3}) P_{M_3}\partial_x (u_{M_1} \cdot v_{M_2})\right\}\right|\right|_{X^{b_1}_{\lambda,M_3}}$$

Let $\gamma : \mathbb{R} \to [0; 1]$ be a smooth partition of unity, satisfying supp$\gamma \subset [-1; 1]$ and

$$\forall x \in \mathbb{R}, \sum_{m\in\mathbb{Z}} \gamma(x - m) = 1$$

Since $(1 \vee M_1) \lesssim M_2 \sim M_3$, we have

$$\chi_{M_3^{-1}}(t - t_{M_3}) = \sum_{|m|,|n|\leqslant 100} \chi_{M_3^{-1}}(t - t_{M_3})\gamma_{M_2^{-1}}(t - t_{M_3} - M_2^{-1}m) \cdot \gamma_{(1\vee M_1)^{-1}}(t - t_{M_3} - M_2^{-1}m - (1 \vee M_1)^{-1}n)$$

Since we take the supremum in $m$ and $n$, without loss of generality, we can assume $m = n = 0$. Thus, if we define

$$f_1^{(1\vee M_1)} := \chi_{(1\vee M_1)}(\tau - \omega)\mathcal{F}\left(\gamma_{(1\vee M_1)^{-1}}(t - t_{M_3})u_{M_1}\right) \text{ and } f_1^{K_1} := \rho_{K_1}(\tau - \omega)\mathcal{F}\left(\gamma_{(1\vee M_1)^{-1}}(t - t_{M_3})u_{M_1}\right), \text{ if } K_1 > (1 \vee M_1) \tag{3.7.2}$$

and as well for $v$

$$f_2^{M_2} := \chi_{M_2}(\tau - \omega)\mathcal{F}\left(\gamma_{M_2^{-1}}(t - t_{M_3})v_{M_2}\right) \text{ and } f_2^{K_2} := \rho_{K_2}(\tau - \omega)\mathcal{F}\left(\gamma_{M_2^{-1}}(t - t_{M_3})v_{M_2}\right), \text{ if } K_2 > M_2 \tag{3.7.3}$$

by splitting the term in the left-hand side according to its modulations, we then get

$$\begin{aligned}
&||P_{M_3}\partial_x\left(u_{M_1}\cdot v_{M_2}\right)||_{N^{b_1}_{\lambda,M_3}}\\
&\quad\lesssim \sup_{t_{M_3}\in\mathbb{R}}\sum_{K_1\geqslant(1\vee M_1)}\sum_{K_2\geqslant M_2}\left|\left|(\tau-\omega+iM_3)^{-1}p\cdot\mathcal{F}\left\{P_{M_3}\partial_x\mathcal{F}^{-1}\left(f_1^{K_1}\star f_2^{K_2}\right)\right\}\right|\right|_{X_{\lambda,M_3}}\\
&\qquad\qquad\qquad\qquad = \sup_{t_{M_3}\in\mathbb{R}}\sum_{K_1\geqslant(1\vee M_1)}\sum_{K_2\geqslant M_2}\sum_{K_3\geqslant 1}K_3^{1/2}\beta^{b_1}_{M_3,K_3}\\
&\qquad\qquad\qquad\qquad\quad\cdot\left|\left|(\tau-\omega+iM_3)^{-1}p\cdot\rho_{K_3}(\tau-\omega)\mathcal{F}\left\{P_{M_3}\partial_x\mathcal{F}^{-1}\left(f_1^{K_1}\star f_2^{K_2}\right)\right\}\right|\right|_{L^2}
\end{aligned}$$

Let us start with the modulations $K_3<M_3$ : the first factor in the previous norm allows us to gain a factor $(M_3\vee K_3)^{-1}$ which makes up for the derivative, thus

$$\begin{aligned}
&\sum_{1\leqslant K_3<M_3}K_3^{1/2}\left|\left|(\tau-\omega+iM_3)^{-1}p\cdot\rho_{K_3}(\tau-\omega)\mathcal{F}\left\{P_{M_3}\partial_x\mathcal{F}^{-1}\left(f_1^{K_1}\star f_2^{K_2}\right)\right\}\right|\right|_{L^2}\\
&\qquad\qquad\qquad\qquad\lesssim\sum_{1\leqslant K_3<M_3}K_3^{1/2}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant M_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}
\end{aligned}$$

and using that $\displaystyle\sum_{1\leqslant K_3<M_3}K_3^{1/2}\lesssim M_3^{1/2}$ we get that the previous sum is controlled with

$$M_3^{1/2}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant M_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}$$

Proceeding as well for the modulations $K_3\geqslant M_3$ and choosing a factor $K_3^{-1}$ instead of $M_3^{-1}$, we get now

$$\begin{aligned}
&\sum_{K_3\geqslant M_3}K_3^{1/2}\beta^{b_1}_{M_3,K_3}\left|\left|(\tau-\omega+iM_3)^{-1}p\cdot\rho_{K_3}(\tau-\omega)\mathcal{F}\left\{P_{M_3}\partial_x\mathcal{F}^{-1}\left(f_1^{K_1}\star f_2^{K_2}\right)\right\}\right|\right|_{L^2}\\
&\qquad\qquad\lesssim M_3\sum_{K_3\geqslant M_3}K_3^{-1/2}\beta^{b_1}_{M_3,K_3}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}
\end{aligned}$$

In particular, the first sum over the modulations $K_3<M_3$ is controlled by the first term in the second sum over the modulations $K_3\geqslant M_3$.

Finally, it suffices to show that $\forall K_i\geqslant(1\vee M_i)$, $i=1,2$,

$$\begin{aligned}
&M_3\sum_{K_3\geqslant M_3}K_3^{-1/2}\beta^{b_1}_{M_3,K_3}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\\
&\qquad\qquad\lesssim M_1^{1/2}\left(K_1^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\right)\left(K_2^{1/2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}\right)\qquad(3.7.4)
\end{aligned}$$

Indeed, combining all the previous estimates, summing over $K_i\geqslant(1\vee M_i)$ and using the definitions of $f_i^{K_i}$ (3.7.2), (3.7.3), the left-hand side of (3.7.1) is controlled by

$$M_1^{1/2}\left(\sum_{K_1\geqslant(1\vee M_1)}K_1^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\right)\left(\sum_{K_2\geqslant M_2}K_2^{1/2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}\right)$$

The first sum is

$$(1\vee M_1)^{1/2}\left|\left|\chi_{(1\vee M_1)}(\tau-\omega)\mathcal{F}\left(\gamma_{(1\vee M_1)^{-1}}(t-t_{M_3})u_{M_1}\right)\right|\right|_{L^2}$$
$$+\sum_{K_1>(1\vee M_1)}K_1^{1/2}\left|\left|p\cdot\rho_{K_1}(\tau-\omega)\mathcal{F}\left\{\gamma_{(1\vee M_1)^{-1}}(t-t_{M_3})u_{M_1}\right\}\right|\right|_{L^2}$$

As $\chi\equiv 1$ on supp$\gamma$, we have

$$\gamma_{(1\vee M_1)^{-1}}(t-t_{M_3})=\gamma_{(1\vee M_1)^{-1}}(t-t_{M_3})\chi_{(1\vee M_1)^{-1}}(t-t_{M_3})$$

so this term is controlled by $||u_{M_1}||_{F^0_{\lambda,M_1}}$ thanks to (3.4.10) and (3.4.11) with

$$f=\mathcal{F}\left\{\chi_{(1\vee M_1)^{-1}}(t-t_{M_3})u_{M_1}\right\}$$

and $K_0=(1\vee M_1)$.

We can similarly bound the second sum by $||v_{M_2}||_{F^0_{\lambda,M_2}}$.

For now, we have established some estimates on expressions in the form $\int f_1\star f_2\cdot f_3$. Thus we first have to express $p\cdot f_1\star f_2$ according to $(p\cdot f_1)$ and $(p\cdot f_2)$. So, using the localizations in $|\xi_i|$ and the relation between the $M_i$, we can estimate

$$\begin{aligned}p(\xi_1+\xi_2,q_1+q_2)&\sim 1+\frac{|q_1+q_2|}{(\xi_1+\xi_2)^2}\\&\lesssim 1+\frac{|q_2|}{\xi_2^2}+\frac{|\xi_1|\langle\xi_1\rangle}{(\xi_1+\xi_2)^2}\cdot\frac{|q_1|}{|\xi_1|\langle\xi_1\rangle}\\&\lesssim p(\xi_2,q_2)+\frac{M_1(1\vee M_1)}{M_3^2}p(\xi_1,q_1)\end{aligned}\qquad(3.7.5)$$

We then treat separately the low and high frequency cases.
**Case 1 :** If $M_1\leqslant 1$.
We use the previous estimate to get

$$\begin{aligned}&\left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\\&\lesssim\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot f_1^{K_1}\star(p\cdot f_2^{K_2})\right|\right|_{L^2}+\frac{M_1(1\vee M_1)}{M_3^2}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot(p\cdot f_1^{K_1})\star f_2^{K_2}\right|\right|_{L^2}\\&=I+II\end{aligned}$$

To treat $I$, we use (3.6.30) :

$$I\lesssim(1\vee M_1)M_1^{1/2}K_{min}^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}$$

Using that $K_2\geqslant M_2\sim M_3$, we obtain

$$I\lesssim(K_1K_2)^{1/2}M_1^{1/2}M_3^{-1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}$$

Next, as we can exchange the roles played by $f_1$ and $f_2$ in (3.6.30), we can also apply this estimate to control $II$ :

$$II\lesssim\frac{M_1(1\vee M_1)}{M_3^2}(1\vee M_2)M_1^{1/2}K_{min}^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}$$

Using that $M_1 \leqslant 1 \leqslant M_3 \sim M_2$, we directly get

$$II \lesssim M_1^{3/2} M_3^{-3/2} (K_1 K_2)^{1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2} \left\|p \cdot f_2^{K_2}\right\|_{L^2}$$

Finally

$$\left\|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right\|_{L^2} \lesssim M_1^{1/2} M_3^{-1/2} \cdot K_1^{1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2} K_2^{1/2} \left\|p \cdot f_2^{K_2}\right\|_{L^2}$$

so after summing

$$M_3 \sum_{K_3 \geqslant M_3} K_3^{-1/2} \beta_{M_3,K_3}^{b_1} \left\|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right\|_{L^2}$$
$$\lesssim M_1^{1/2} \cdot K_1^{1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2} K_2^{1/2} \left\|p \cdot f_2^{K_2}\right\|_{L^2}$$

since

$$\sum_{K_3 \geqslant M_3} K_3^{-1/2} \beta_{M_3,K_3}^{b_1} \lesssim M_3^{-1/2}$$

This is (3.7.4) in that case.
**Case 2 :** If $M_1 > 1$.
It is still sufficient to use (3.7.5) if $K_3$ is large enough .

Indeed, let us split the sum over $K_3$ into two parts, depending on whether $K_3 \geqslant M_1^2 M_3$ or $M_3 \leqslant K_3 \leqslant M_1^2 M_3$.
**Case 2.1 :** If $K_3 \geqslant M_1^2 M_3$.
We proceed as in the case $M_1 \leqslant 1$ to get

$$\left\|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right\|_{L^2}$$
$$\lesssim \left\|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot f_1^{K_1} \star (p \cdot f_2^{K_2})\right\|_{L^2} + \frac{M_1(1 \vee M_1)}{M_3^2} \left\|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot (p \cdot f_1^{K_1}) \star f_2^{K_2}\right\|_{L^2}$$
$$= I + II$$

As previously,

$$I \lesssim M_1^{3/2} K_{min}^{1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2} \left\|p \cdot f_2^{K_2}\right\|_{L^2} \lesssim M_1^{3/2} (K_1 K_2)^{1/2} M_2^{-1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2} \left\|p \cdot f_2^{K_2}\right\|_{L^2}$$

As for $II$, we have again

$$II \lesssim M_1^{5/2} M_2 M_3^{-2} K_{min}^{1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2} \left\|p \cdot f_2^{K_2}\right\|_{L^2}$$
$$\lesssim M_1^{3/2} (K_1 K_2)^{1/2} M_2^{-1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2} \left\|p \cdot f_2^{K_2}\right\|_{L^2}$$

It remains to sum for the modulations $K_3 \geqslant M_1^2 M_3$ :

$$M_3 \sum_{K_3 \geqslant M_1^2 M_3} K_3^{-1/2} \beta_{M_3,K_3}^{b_1} \left\|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right\|_{L^2}$$
$$\lesssim M_1^{1/2} \left(K_1^{1/2} \left\|p \cdot f_1^{K_1}\right\|_{L^2}\right) \left(K_2^{1/2} \left\|p \cdot f_2^{K_2}\right\|_{L^2}\right) \quad (3.7.6)$$

since
$$\sum_{K_3\geqslant M_1^2M_3} K_3^{-1/2}\beta_{M_3,K_3}^{b_1} \lesssim M_1^{-1}M_3^{-1/2}\beta_{M_3,M_1^2M_3}^{b_1}$$
and for $M_1>1$, we have $M_1^2M_3<M_3^3$ so $\beta_{M_3,M_1^2M_3}=1$.

**Case 2.2 :** If $M_3\leqslant K_3\leqslant M_1^2M_3$.

We improve (3.7.5) using the resonant function (cf. (3.6.9)). Observe that, since $\Omega(\zeta_1,\zeta_2,\zeta_3)$ and the hyperplane $\zeta_1+\zeta_2+\zeta_3=0$ are invariant under permutation, we have
$$\left|\frac{q_1+q_2}{\xi_1+\xi_2}-\frac{q_2}{\xi_2}\right| = \left|\frac{\xi_2}{\xi_1(\xi_1+\xi_2)}\Omega(-\zeta_1-\zeta_2,\zeta_2,\zeta_1)+3\xi_1^2\right|^{1/2}$$
Since $\mathrm{supp} f_i\subset D_{\lambda,M_i,\leqslant K_i}$ and $\int f_1\star f_2\cdot f_3\neq 0\Rightarrow |\Omega|\lesssim K_{max}$, we deduce the bound
$$p(\xi_1+\xi_2,q_1+q_2)\lesssim 1+\frac{|q_1+q_2|}{|\xi_1+\xi_2|^2}\lesssim p(\xi_2,q_2)+M_1^{1/2}M_3^{-2}K_{max}^{1/2} \tag{3.7.7}$$
Therefore, we have the bound
$$\begin{aligned}&\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\\ &\quad\lesssim \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot f_1^{K_1}\star (p\cdot f_2^{K_2})\right|\right|_{L^2}+M_1^{-1/2}M_3^{-1}K_{max}^{1/2}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\end{aligned}$$
as $M_1\lesssim M_3$.

To treat those terms, we distinguish the cases of corollary 3.6.9.

Case 2.1 (a) : If $K_{max}\lesssim M_1M_2M_3$. In that case we estimate $K_{max}^{1/2}$ in the second term and then apply (3.6.27) to both terms to get the bound
$$\begin{aligned}&M_3\sum_{K_3=M_3}^{M_1^2M_3}K_3^{-1/2}\beta_{M_3,K_3}^{b_1}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\\ &\qquad\lesssim \ln(M_1)\,M_1^{-1/2}\cdot(K_1K_2)^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}\end{aligned}$$
Case 2.2 (b)&(c) : If $K_{max}\gtrsim M_1M_2M_3$. Then we lose the factor $K_{max}^{1/2}$ in the first term and use (3.6.7) for both terms with the indices corresponding to $K_{min}$ and $K_{med}$, getting the final bound
$$\begin{aligned}&M_3\sum_{K_3=M_3}^{M_1^2M_3}K_3^{-1/2}\beta_{M_3,K_3}^{b_1}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot p\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\\ &\qquad\lesssim \ln(M_1)\cdot(K_1K_2)^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}\end{aligned}$$
□

**Lemma 3.7.2 (*High* × *High* → *Low*)**

*Let $M_1,M_2,M_3\in 2^{\mathbb{Z}}$ with $M_1\sim M_2\gtrsim(1\vee M_3)$, and $b_1\in[0;1/2[$. Then for $u_{M_1}\in N^0_{\lambda,M_1}$ and $v_{M_2}\in N^0_{\lambda,M_2}$, we have*
$$||P_{M_3}\partial_x(u_{M_1}\cdot v_{M_2})||_{N^{b_1}_{\lambda,M_3}}\lesssim M_2^{3/2+4b_1}(1\vee M_3)^{-1}\,||u_{M_1}||_{F^0_{\lambda,M_1}}\,||v_{M_2}||_{F^0_{\lambda,M_2}} \tag{3.7.8}$$

**Proof :**
We proceed similarly to the previous lemma, but this time the norm on the left-hand side only controls functions on time intervals of size $(1 \vee M_3)^{-1}$ whereas the norms on the right-hand side require a control for time intervals of size $M_2^{-1}$. Thus will cut the time intervals in smaller pieces.

To do so, we take $\gamma$ as in the previous lemma. Since now $M_1 \sim M_2 \gtrsim (1 \vee M_3)$, we can write

$$\chi_{(1\vee M_3)^{-1}}(t - t_{M_3}) = \sum_{|m| \lesssim M_2(1\vee M_3)^{-1}} \sum_{|n|\lesssim 100} \chi_{(1\vee M_3)^{-1}}(t - t_{M_3})\gamma_{M_2}(t - t_{M_3} - M_2^{-1}m) \cdot \gamma_{M_1}(t - t_{M_3} - M_2^{-1}m - M_1^{-1}n)$$

As previously, without loss of generality, we can assume $m = n = 0$, and defining

$$f_1 := \mathcal{F}\left\{\gamma\left(M_1(t - t_{M_3})\right) u_{M_1}\right\}$$

and

$$f_2 := \mathcal{F}\left\{\gamma\left(M_2(t - t_{M_3})\right) v_{M_2}\right\}$$

it then suffices to prove that $\forall K_i \geqslant (1 \vee M_i)$ :

$$M_2(1 \vee M_3)^{-1} \cdot M_3 \sum_{K_3 \geqslant (1\vee M_3)} K_3^{-1/2}\beta^{b_1}_{M_3,K_3} \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2} \lesssim M_2^2(1 \vee M_3)^{-1}K_1^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} K_2^{1/2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \quad (3.7.9)$$

where we have denoted

$$f_i^{M_2} := \chi_{M_i}(\tau - \omega)f_i \text{ and } f_i^{K_i} := \rho_{K_i}(\tau - \omega)f_i,\ K_i > M_i$$

As previously, we need to estimate $p(\xi_1 + \xi_2, q_1 + q_2)$ with respect to $p(\xi_1, q_1)$ and $p(\xi_2, q_2)$ :

$$\begin{aligned} p(\xi_1 + \xi_2, q_1 + q_2) &\lesssim 1 + \frac{|q_1 + q_2|}{|\xi_1 + \xi_2|\langle\xi_1 + \xi_2\rangle} \\ &\lesssim 1 + \frac{|\xi_1|\langle\xi_1\rangle}{|\xi_1 + \xi_2|\langle\xi_1 + \xi_2\rangle}\frac{|q_1|}{|\xi_1|\langle\xi_1\rangle} + \frac{|\xi_2|\langle\xi_2\rangle}{|\xi_1 + \xi_2|\langle\xi_1 + \xi_2\rangle}\frac{|q_2|}{|\xi_2|\langle\xi_2\rangle} \\ &\lesssim M_2^2 M_3^{-1}(1 \vee M_3)^{-1}\left(p(\xi_1, q_1) + p(\xi_2, q_2)\right) \quad (3.7.10) \end{aligned}$$

Just as before, we distinguish several cases.
**Case 1.1 :** If $M_3 \leqslant 1$ and $K_3 \geqslant M_2^5$ :
We use (3.7.10), so that the left-hand side of (3.7.9) is controlled with

$$M_2^3 \sum_{K_3 \geqslant M_2^5} K_3^{b_1-1/2} \left\{\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot (p \cdot f_1^{K_1}) \star f_2^{K_2}\right|\right|_{L^2} + \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot f_1^{K_1} \star (p \cdot f_2^{K_2})\right|\right|_{L^2}\right\}$$

Using (3.6.30) and that $M_1 \sim M_2 \geqslant 1$ and $K_1, K_2 \gtrsim M_2$, we get the bound

$$\sum_{K_3 \geqslant M_2^5} K_3^{b_1-1/2} M_2^3 \cdot M_2 M_3^{1/2} K_{min}^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \lesssim M_2^{1+5b_1} M_3^{1/2} \cdot (K_1K_2)^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

which suffices for (3.7.9).
**Case 1.2 :** If $M_3 \leqslant 1$ and $1 \leqslant K_3 \leqslant M_2^5$ :
We improve the control on $p$ in this regime by using $\Omega$ as in (3.7.7). We get in this case

$$\left|\frac{q_1+q_2}{\xi_1+\xi_2} - \frac{q_1}{\xi_1}\right| = \left|\frac{\xi_2}{\xi_1(\xi_1+\xi_2)}\Omega(\zeta_1, -\zeta_1-\zeta_2, \zeta_2) + 3\xi_2^2\right|^{1/2} \lesssim M_2 + M_3^{-1/2}K_{max}^{1/2}$$

from which we deduce

$$p(\xi_1+\xi_2, q_1+q_2) \lesssim M_2 p(\xi_1, q_1) + M_3^{-1/2}K_{max}^{1/2} \tag{3.7.11}$$

Using this estimate, we get the bound

$$M_3M_2^2 \sum_{K_3=1}^{M_2^5} K_3^{b_1-1/2}\Big\{\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot(p\cdot f_1^{K_1})\star f_2^{K_2}\right|\right|_{L^2} + M_3^{-1/2}M_2^{-1}K_{max}^{1/2}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\Big\}$$

Observe that the term within the braces is the same as in case 2.2 of lemma 3.7.1, so we control it the exact same way to get the final bound

$$M_3^{1/2}M_2^{1+5b_1}\cdot(K_1K_2)^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}$$

**Case 2.1 :** If $M_3 \geqslant 1$ and $K_3 \geqslant M_2^4M_3^{-1}$.
We use again (3.7.10) so that the left-hand side of (3.7.9) is controlled with

$$M_2 \sum_{K_3\geqslant M_2^4M_3^{-1}} K_3^{-1/2}\beta_{M_3,K_3}^{b_1}M_2^2M_3^{-2}\Big\{\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot(p\cdot f_1^{K_1})\star f_2^{K_2}\right|\right|_{L^2} + \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot f_1^{K_1}\star(p\cdot f_2^{K_2})\right|\right|_{L^2}\Big\}$$

With (3.6.30) again, we obtain the bound

$$\sum_{K_3\geqslant M_2^4M_3^{-1}} K_3^{-1/2}\beta_{M_3,K_3}^{b_1}M_2^3M_3^{-2}\cdot M_2M_3^{1/2}K_{min}^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}$$
$$\lesssim M_2^{3/2+4b_1}M_3^{-1-4b_1}\cdot(K_1K_2)^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}K_2^{1/2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}$$

**Case 2.2 :** If $M_3 \geqslant 1$ and $M_3 \leqslant K_3 \leqslant M_2^4M_3^{-1}$.
(3.7.11) becomes in this case

$$p(\xi_1+\xi_2, q_1+q_2) \lesssim M_3^{-1}M_2 p(\xi_1, q_1) + M_3^{-3/2}K_{max}^{1/2} \tag{3.7.12}$$

So the use of (3.7.12) allows us to bound the left-hand side of (3.7.9) with

$$M_2^2M_3^{-1}\sum_{K_3=M_3}^{M_2^4M_3^{-1}} K_3^{-1/2}\beta_{M_3,K_3}^{b_1}\Big\{\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot(p\cdot f_1^{K_1})\star f_2^{K_2}\right|\right|_{L^2} + M_3^{-1/2}M_2^{-1}K_{max}^{1/2}\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}}\cdot f_1^{K_1}\star f_2^{K_2}\right|\right|_{L^2}\Big\}$$

Proceeding similarly to the previous cases, we finally obtain the bound

$$M_2^{1+4b_1}M_3^{-1-4b_1}\cdot(K_1K_2)^{1/2}\left|\left|p\cdot f_1^{K_1}\right|\right|_{L^2}K_2^{1/2}\left|\left|p\cdot f_2^{K_2}\right|\right|_{L^2}$$

□

**Lemma 3.7.3 (*Low* × *Low* → *Low*)**

*Let $M_1, M_2, M_3 \in 2^{-\mathbb{Z}}$ and $b_1 \in [0;1/2[$. Then for $u_{M_1} \in F^0_{\lambda,M_1}$ and $v_{M_2} \in F^0_{\lambda,M_2}$ we have*

$$||P_{M_3}\partial_x (u_{M_1} \cdot v_{M_2})||_{N^{b_1}_{\lambda,M_3}} \lesssim (M_1M_2M_3)^{1/2} ||u_{M_1}||_{F^0_{\lambda,M_1}} ||v_{M_2}||_{F^0_{\lambda,M_2}} \tag{3.7.13}$$

**Proof :**
As in the previous lemmas, it is enough to prove that $\forall K_1, K_2 \geqslant 1$,

$$M_3 \sum_{K_3 \geqslant 1} K_3^{-1/2} \beta^{b_1}_{M_3,K_3} \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2} \lesssim K_1^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} K_2^{1/2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \tag{3.7.14}$$

By symmetry, we may assume $M_1 \leqslant M_2$, so similarly to (3.7.10), we have in this case

$$p(\xi_1+\xi_2, q_1+q_2) \lesssim M_2M_3^{-1}\left(p(\xi_1,q_1)+p(\xi_2,q_2)\right)$$

It then suffices to use (3.6.30) along with the previous bound to get (3.7.14) :

$$\begin{aligned} M_3 \sum_{K_3 \geqslant 1} K_3^{-1/2} \beta^{b_1}_{M_3,K_3} &\left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2} \\ &\lesssim M_2 \sum_{K_3 \geqslant 1} K_3^{b_1-1/2} M_{min}^{1/2} K_{min}^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \\ &\lesssim (M_1M_2M_3)^{1/2} \cdot (K_1K_2)^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \end{aligned}$$

□

**Proposition 3.7.4**

*Let $T \in ]0;1]$, $\alpha \geqslant 1$ and $b_1 \in [0;1/8]$. Then for $u, v \in \mathbf{F}^{\alpha,0}_\lambda(T)$ we have*

$$||\partial_x(uv)||_{\mathbf{N}^{\alpha,b_1}_\lambda(T)} \lesssim ||u||_{\mathbf{F}^{\alpha,0}_\lambda(T)} ||v||_{\mathbf{F}^{1,0}_\lambda(T)} + ||u||_{\mathbf{F}^{1,0}_\lambda(T)} ||v||_{\mathbf{F}^{\alpha,0}_\lambda(T)} \tag{3.7.15}$$

**Proof :**
For $M_1 \in 2^{\mathbb{Z}}$, let us choose an extension $u_{M_1} \in F^0_{\lambda,M_1}$ of $P_{M_1}u$ satisfying

$$||u_{M_1}||_{F^0_{\lambda,M_1}} \leqslant 2\, ||P_{M_1}u||_{F^0_{\lambda,M_1}(T)}$$

and let us define $v_{M_2}$ analogously.

Using the definition of $\mathbf{F}^{\alpha,b_1}_\lambda(T)$ (3.4.2) and $\mathbf{N}^{\alpha,b_1}_\lambda(T)$ (3.4.3), it then suffices to show that

$$\begin{aligned} \sum_{M_1,M_2,M_3} (1 \vee M_3)^{2\alpha} &||P_{M_3}\partial_x(u_{M_1} \cdot v_{M_2})||^2_{N^{b_1}_{\lambda,M_3}} \\ &\lesssim \sum_{M_1,M_2} \Big\{(1 \vee M_1)^{2\alpha} ||u_{M_1}||^2_{F^0_{\lambda,M_1}} (1 \vee M_2)^2 ||v_{M_2}||^2_{F^0_{\lambda,M_2}} \\ &\qquad + (1 \vee M_1)^2 ||u_{M_1}||^2_{F^0_{\lambda,M_1}} (1 \vee M_2)^{2\alpha} ||v_{M_2}||^2_{F^0_{\lambda,M_2}}\Big\} \end{aligned} \tag{3.7.16}$$

Since the left-hand side of (3.7.15) is symmetrical in $u$ and $v$, we can assume $M_1 \leqslant M_2$. Then we can decompose the left-hand side of (3.7.16) depending on the relation between $M_1, M_2$ and $M_3$ :

$$\sum_{M_1,M_2,M_3>0} (1 \vee M_3)^{2\alpha} \, ||P_{M_3}\partial_x(u_{M_1} \cdot v_{M_2})||^2_{N^{b_1}_{\lambda,M_3}}$$
$$= \sum_{i=1}^{3} \sum_{(M_1,M_2,M_3)\in A_i} (1 \vee M_3)^{2\alpha} \, ||P_{M_3}\partial_x(u_{M_1} \cdot v_{M_2})||^2_{N^{b_1}_{\lambda,M_3}}$$

where

$$\begin{cases} A_1 := \left\{(M_1, M_2, M_3) \in 2^{\mathbb{Z}}, (1 \vee M_1) \lesssim M_2 \sim M_3\right\} \\ A_2 := \left\{(M_1, M_2, M_3) \in 2^{\mathbb{Z}}, (1 \vee M_3) \lesssim M_1 \sim M_2\right\} \\ A_3 := \left\{(M_1, M_2, M_3) \in 2^{\mathbb{Z}}, M_{max} \lesssim 1\right\} \end{cases}$$

Using lemma 3.7.1, the first term is estimated by

$$\sum_{(M_1,M_2,M_3)\in A_1} (1 \vee M_3)^{2\alpha} \, ||P_{M_3}\partial_x(u_{M_1} \cdot v_{M_2})||^2_{N^{b_1}_{\lambda,M_3}}$$
$$\lesssim \sum_{M_2 \gtrsim 1} \sum_{M_1 \lesssim M_2} M_1 (1 \vee M_2)^{2\alpha} \, ||u_{M_1}||^2_{F^0_{\lambda,M_1}} \, ||v_{M_2}||^2_{F^0_{\lambda,M_2}}$$

which suffices for (3.7.16). For the second term, the use of lemma 3.7.2 provides the bound

$$\sum_{(M_1,M_2,M_3)\in A_2} (1 \vee M_3)^{2\alpha} \, ||P_{M_3}\partial_x(u_{M_1} \cdot v_{M_2})||^2_{N^{b_1}_{\lambda,M_3}}$$
$$\lesssim \sum_{M_2 \gtrsim 1} \sum_{M_1 \sim M_2} M_2^{3+8b_1+2(\alpha-1)} \, ||u_{M_1}||^2_{F^0_{\lambda,M_1}} \, ||v_{M_2}||^2_{F^0_{\lambda,M_2}}$$

which is enough for (3.7.16) since $b_1 \in [0; 1/8]$. Finally, lemma 3.7.3 allows us to control the last term by

$$\sum_{(M_1,M_2,M_3)\in A_3} (1 \vee M_3)^{2\alpha} \, ||P_{M_3}\partial_x(u_{M_1} \cdot v_{M_2})||^2_{N^{b_1}_{\lambda,M_3}}$$
$$\lesssim \sum_{M_1 \in 2^{-\mathbb{N}}} \sum_{M_2 \in 2^{-\mathbb{N}}} M_1 M_2 \, ||u_{M_1}||^2_{F^0_{\lambda,M_1}} \, ||v_{M_2}||^2_{F^0_{\lambda,M_2}}$$

which concludes the proof of the bilinear estimate.

□

### 3.7.2 For the difference equation

The end of this section is devoted to treating (3.1.15). Let $b_1 \in [0; 1/2[$.

We begin with the low frequency interactions :

**Lemma 3.7.5 (*Low* × *Low* → *Low*)**

*Let $M_1, M_2, M_3 \in 2^{-\mathbb{Z}}$. Then for $u_{M_1} \in \overline{F^0_{\lambda,M_1}}$ and $v_{M_2} \in F^0_{\lambda,M_2}$, we have*

$$||P_{M_3}\partial_x \left(u_{M_1} \cdot v_{M_2}\right)||_{\overline{N^{b_1}_{\lambda,M_3}}} \lesssim M_3 M_{min}^{1/2} \, ||u_{M_1}||_{\overline{F^{b_1}_{\lambda,M_1}}} \, ||v_{M_2}||_{F^{b_1}_{\lambda,M_2}}$$

**Proof :**
Proceeding as for the previous lemmas, it suffices to prove that for all $K_1, K_2 \geqslant 1$ and $f_i^{K_i} : D_{\lambda,M_i,\leqslant K_i} \to \mathbb{R}_+$,

$$M_3 \sum_{K_3 \geqslant 1} K_3^{-1/2} \beta M_3, K_3{}^{b_1} \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2}$$
$$\lesssim M_3 M_{min}^{1/2} \cdot (K_1K_2)^{1/2} \left|\left|f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

This follows directly from (3.6.30).

□

**Lemma 3.7.6 (*High × High → Low*)**

*Let $M_1, M_2, M_3 \in 2^{\mathbb{Z}}$ with $M_1 \sim M_2 \gtrsim (1 \vee M_3)$. Then for $u_{M_1} \in \overline{F^0_{\lambda,M_1}}$ and $v_{M_2} \in F^0_{\lambda,M_2}$, we have*

$$||P_{M_3}\partial_x (u_{M_1} \cdot v_{M_2})||_{\overline{N^{b_1}_{\lambda,M_3}}} \lesssim (1 \wedge M_3)^{3/2} M_2 \, ||u_{M_1}||_{\overline{F^{b_1}_{\lambda,M_1}}} \, ||v_{M_2}||_{F^{b_1}_{\lambda,M_2}}$$

**Proof :**
Following the proof of lemma 3.7.2, it is enough to prove that for all $K_i \geqslant (1 \vee M_i)$ and $f_i^{K_i} : D_{\lambda,M_i,\leqslant K_i} \to \mathbb{R}_+$,

$$M_3 M_2 (1 \vee M_3)^{-1} \sum_{K_3 \geqslant (1\vee M_3)} K_3^{-1/2} \beta M_3, K_3{}^{b_1} \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2}$$
$$\lesssim (1 \wedge M_3)^{3/2} M_2 \cdot (K_1K_2)^{1/2} \left|\left|f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

This is a consequence of (3.6.7).

□

It remains to treat the interaction between low and high frequencies. Since $u$ and $v$ do not play a symetric role anymore, we have to distinguih which one has the low frequency part.

**Lemma 3.7.7 (*Low × High → High*)**

*Let $(1 \vee M_1) \lesssim M_2 \sim M_3$ and $u_{M_1} \in \overline{F^0_{\lambda,M_1}}$, $v_{M_2} \in F^0_{\lambda,M_2}$. Then*

$$||P_{M_3}\partial_x (u_{M_1} \cdot v_{M_2})||_{\overline{N^{b_1}_{\lambda,M_3}}} \lesssim M_1^{1/2} (1 \vee M_1)^{1/4} M_2^{1/4} \, ||u_{M_1}||_{\overline{F^0_{\lambda,M_1}}} \, ||v_{M_2}||_{F^0_{\lambda,M_2}}$$

**Proof :**
Foolowing the proof of lemma 3.7.1, it suffices to prove that for all $K_i \geqslant (1 \vee M_i)$ and $f_i^{K_i} : D_{\lambda,M_i,\leqslant K_i} \to \mathbb{R}_+$,

$$M_3 \sum_{K_3 \geqslant M_3} K_3^{-1/2} \beta M_3, K_3{}^{b_1} \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2}$$
$$\lesssim M_1^{1/2} (1 \vee M_1)^{1/4} M_2^{1/4} \cdot (K_1K_2)^{1/2} \left|\left|f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

Again, this follows from using (3.6.7).

□

**Lemma 3.7.8 (*High* × *Low* → *High*)**

*Let* $(1 \vee M_2) \lesssim M_1 \sim M_3$ *and* $u_{M_1} \in \overline{F^0_{\lambda,M_1}}$, $v_{M_2} \in F^0_{\lambda,M_2}$. *Then*

$$||P_{M_3}\partial_x(u_{M_1} \cdot v_{M_2})||_{\overline{N^0_{\lambda,M_3}}} \lesssim (1 \vee M_2)\, ||u_{M_1}||_{\overline{F^0_{\lambda,M_1}}}\, ||v_{M_2}||_{F^0_{\lambda,M_2}}$$

**Proof :**
As previously, it is enough to prove

$$M_3 \sum_{K_3 \geqslant M_3} K_3^{-1/2} \left|\left|\mathbb{1}_{D_{\lambda,M_3,\leqslant K_3}} \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2} \lesssim (1 \vee M_2) \cdot (K_1K_2)^{1/2} \left|\left|f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

for $K_i \geqslant (1 \vee M_i)$ and $f_i^{K_i} : D_{\lambda,M_i,\leqslant K_i} \to \mathbb{R}_+$.
Following the proof of lemma 3.7.1,we distinguish several cases.
**Case 1 :** If $M_2 \leqslant 1$.
This is a consequence of (3.6.30).
**Case 2 :** If $M_2 \geqslant 1$.
We split the sum over $K_3$ into two parts. The high modulations part $K_3 \geqslant M_2M_3$ is treated again with (3.6.30), whereas for the sum over the modulations $M_3 \leqslant K_3 \leqslant M_2M_3$ is controlled by using (3.6.28) (which is the worst case of corollary 3.6.9).

□

We finally combine the previous estimates to get

**Proposition 3.7.9**

*Let* $T \in ]0;1]$, $b_1 \in [0;1/2[$ *and* $u \in \overline{\mathbf{F}_\lambda}^0(T)$, $v \in \mathbf{F}_\lambda^{1,0}(T)$. *Then*

$$||\partial_x(uv)||_{\overline{\mathbf{N}_\lambda}^{b_1}(T)} \lesssim ||u||_{\overline{\mathbf{F}_\lambda}^0(T)}\, ||v||_{\mathbf{F}_\lambda^{1,0}(T)} \tag{3.7.17}$$

**Proof :**
First, for $M_1, M_2 \in 2^{\mathbb{Z}}$, we fix an extension $u_{M_1} \in \overline{F^0_{\lambda,M_1}}$ of $P_{M_1}u$ to $\mathbb{R}$ satisfying

$$||u_{M_1}||_{\overline{F^0_{\lambda,M_1}}} \leqslant 2\, ||P_{M_1}u||_{\overline{F^0_{\lambda,M_1}}(T)}$$

and similarly for $v_{M_2}$.
Using the definition of $\overline{\mathbf{F}_\lambda}^0$ (3.4.7) and $\overline{\mathbf{N}_\lambda}^{b_1}$ (3.4.8), it then suffices to show that

$$\sum_{M_1,M_2,M_3 \in 2^{\mathbb{Z}}} ||\partial_x(u_{M_1} \cdot v_{M_2})||^2_{\overline{N^{b_1}_{\lambda,M_3}}(T)} \lesssim \sum_{M_1,M_2 \in 2^{\mathbb{Z}}} ||u_{M_1}||^2_{\overline{F^0_{\lambda,M_1}}(T)} (1 \vee M_2)^2\, ||v_{M_2}||^2_{F^0_{\lambda,M_2}(T)} \tag{3.7.18}$$

As in the proof of proposition 3.7.4, we separate 4 cases, so it suffices to show that for $i \in \{1,2,3,4\}$,

$$\sum_{(M_1,M_2,M_3)\in B_i} ||\partial_x(u_{M_1}\cdot v_{M_2})||^2_{\overline{N^{b_1}_{\lambda,M_3}}(T)} \lesssim \sum_{M_1,M_2\in 2^{\mathbb{Z}}} ||u_{M_1}||^2_{\overline{F^0_{\lambda,M_1}}(T)} (1\vee M_2)^2 \, ||v_{M_2}||^2_{F^0_{\lambda,M_2}(T)}$$

with

$$\begin{cases} B_1 := \left\{(M_1,M_2,M_3)\in 2^{-\mathbb{Z}}\right\} \\ B_2 := \left\{(M_1,M_2,M_3)\in 2^{\mathbb{Z}},\ M_1\sim M_2 \gtrsim (1\vee M_3)\right\} \\ B_3 := \left\{(M_1,M_2,M_3)\in 2^{\mathbb{Z}},\ M_2\sim M_3 \gtrsim (1\vee M_1)\right\} \\ B_4 := \left\{(M_1,M_2,M_3)\in 2^{\mathbb{Z}},\ M_1\sim M_3 \gtrsim (1\vee M_2)\right\} \end{cases}$$

This follows from lemmas 3.7.5, 3.7.6, 3.7.7, and 3.7.8 respectively.

□

## 3.8 Energy estimates

In this section we prove the energy estimates (3.1.13) and (3.1.16). As the nonlinear term is expressed as a bilinear form, we will need some control on trilinear form to deal with the energy estimate :

**Lemma 3.8.1**

*Let $T \in [0;1[$, $M_1, M_2, M_3 \in 2^{\mathbb{Z}}$ with $M_{max} \geqslant 1$, and $b_1 \in [0;1/8]$. Then for $u_i \in \overline{F^{b_1}_{\lambda,M_i}}(T)$, $i \in \{1,2,3\}$, with one of them in $F^{b_1}_{\lambda,M_i}(T)$ (in order for the integral to converge), we have*

$$\left|\int_{[0,T]\times\mathbb{R}\times\mathbb{T}_\lambda} u_1u_2u_3 \mathrm{d}t\mathrm{d}x\mathrm{d}y\right| \lesssim \Lambda_{b_1}(M_{min}, M_{max}) \prod_{i=1}^{3} ||u_i||_{\overline{F^{b_1}_{\lambda,M_i}}(T)} \qquad (3.8.1)$$

*where*

$$\Lambda_{b_1}(X,Y) = \left(X\wedge X^{-1}\right)^{1/2} + \left(\frac{(1\vee X)}{Y}\right)^{2b_1} \qquad (3.8.2)$$

**Proof :**
Using the symmetry property (3.6.11), we may assume $M_1 \leqslant M_2 \leqslant M_3$. We begin by fixing some extensions $u_{M_i} \in \overline{F^{b_1}_{\lambda,M_i}}$ of $u_i$ to $\mathbb{R}$ satisfying $||u_{M_i}||_{\overline{F^{b_1}_{\lambda,M_i}}} \leqslant 2\,||u_i||_{\overline{F^{b_1}_{\lambda,M_i}}(T)}$.
Let $\gamma : \mathbb{R} \to [0;1]$ be a smooth partition of unity as in the proof of lemma 3.7.1, satisfying now $\mathrm{supp}\gamma \subset [-1;1]$ and

$$\forall t \in \mathbb{R},\ \sum_{n\in\mathbb{Z}} \gamma^3(t-n) = 1 \qquad (3.8.3)$$

We then use $\gamma$ to slice the time interval in pieces of size $M_3^{-1}$ :

$$\int_{[0;T]\times\mathbb{R}\times\mathbb{T}_\lambda} u_1u_2u_3 \mathrm{d}t\mathrm{d}x\mathrm{d}y = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_{1,n}\star f_{2,n}\cdot f_{3,n} \mathrm{d}\tau\mathrm{d}\xi(\mathrm{d}q)_\lambda \qquad (3.8.4)$$

where we define

$$f_{i,n} := \mathcal{F}\left(\gamma(M_3t-n)\mathbb{1}_{[0,T]}u_{M_i}\right)$$

We can divide the set of integers such that the trilinear form is not zero into two subsets

$$\mathcal{A} := \left\{n \in \mathbb{Z},\ \gamma(M_3 t - n)\mathbb{1}_{[0,T]} = \gamma(M_3 t - n)\right\}$$
$$\text{and } \mathcal{B} = \left\{n \in \mathbb{Z} \setminus \mathcal{A},\ \int f_{1,n} \star f_{2,n} \cdot f_{3,n} \neq 0\right\}$$

Let us notice that $\#\mathcal{A} \lesssim M_3$ and $\#\mathcal{B} \leqslant 4$.

Let us start by dealing with the sum over $\mathcal{A}$ :

$$\left|\sum_{n\in\mathcal{A}} \int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_{1,n} \star f_{2,n} \cdot f_{3,n}\right| \lesssim M_3 \sup_{n\in\mathcal{A}} \left|\sum_{K_1,K_2,K_3\geqslant M_3} \int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_{1,n}^{K_1} \star f_{2,n}^{K_2} \cdot f_{3,n}^{K_3}\right|$$

where $f_{i,n}^{K_i}$ is defined as

$$f_{i,n}^{K_i}(\tau,\xi,q) := \rho_{K_i}(\tau - \omega(\xi,q)) f_{i,n}(\tau,\xi,q),\ i = 1,2,3 \text{ if } K_i > M_3 \tag{3.8.5}$$

and

$$f_{i,n}^{M_3}(\tau,\xi,q) := \chi_{M_3}(\tau - \omega(\xi,q)) f_{i,n}(\tau,\xi,q),\ i = 1,2,3$$

Then, we separate the sum into three parts depending on the relations between the $M$'s and the $K$'s as in corollary 3.6.9 :

$$\sum_{K_1,K_2,K_3} \int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_{1,n}^{K_1} \star f_{2,n}^{K_2} \cdot f_{3,n}^{K_3} = \sum_{i=1}^{3} \sum_{(K_1,K_2,K_3)\in A_i} \int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_{1,n}^{K_1} \star f_{2,n}^{K_2} \cdot f_{3,n}^{K_3}$$

with

$$\begin{cases} A_1 := \left\{(K_1,K_2,K_3) \in 2^{\mathbb{N}},\ K_i \geqslant M_3,\ K_{max} \leqslant 10^{-10} M_1 M_2 M_3\right\} \\ A_2 := \left\{(K_1,K_2,K_3) \in 2^{\mathbb{N}},\ K_i \geqslant M_3,\ K_1 = K_{max} \gtrsim M_1 M_2 M_3\right\} \\ A_3 := \left\{(K_1,K_2,K_3) \in 2^{\mathbb{N}},\ K_i \geqslant M_3,\ K_{max} = (K_2 \vee K_3) \gtrsim M_1 M_2 M_3\right\} \end{cases}$$

We treat those terms separately, using the estimates of corollary 3.6.9. Denoting $J_i$ the contributioin of the region $A_i$ in the sum, we have

$$|J_1| \lesssim M_3 \sup_{n\in\mathcal{A}} \sum_{K_1,K_2,K_3\geqslant M_3} \left(M_{min} \wedge M_{min}^{-1}\right)^{1/2} M_{max}^{-1} \prod_{i=1}^{3} K_i^{1/2} \left|\left|f_{i,n}^{K_i}\right|\right|_{L^2}$$
$$\lesssim \left(M_{min} \wedge M_{min}^{-1}\right)^{1/2} \prod_{i=1}^{3} ||u_{M_i}||_{\overline{F^0_{\lambda,M_i}}}$$

after using (3.6.27) and

$$\sup_{n\in\mathcal{A}} \sum_{K_i\geqslant M_3} K_i^{1/2} \left|\left|f_{i,n}^{K_i}\right|\right|_{L^2} \lesssim ||u_{M_i}||_{\overline{F^0_{\lambda,M_i}}} \tag{3.8.6}$$

Indeed, (3.8.6) follows from the definition of $f_{i,n}^{K_i}$ (3.8.5), the fact that $\chi_{(1\vee M_i)^{-1}} \equiv 1$ on the support of $\gamma_{M_3^{-1}}$, and the use of (3.4.10) and (3.4.11).

Proceeding analogously, we get

$$|J_3| \lesssim M_3 \sup_{n\in\mathcal{A}} \sum_{K_1,K_2,K_3\geqslant M_3} M_{max}^{-1} \left(\frac{(1\vee M_{min})}{M_{max}}\right)^{1/4} \prod_{i=1}^{3} K_i^{1/2} \left|\left|f_{i,n}^{K_i}\right|\right|_{L^2}$$
$$\lesssim \left(\frac{(1\vee M_{min})}{M_{max}}\right)^{2b_1} \prod_{i=1}^{3} ||u_i||_{\overline{F^0_{\lambda,M_i}}}$$

by using (3.6.29) and that $b_1 \in [0;1/8]$.

Finally, the last contribution is controlled thanks to (3.6.28), (3.8.6) and the weight $\beta_{M_1,K_1}^{b_1}$ :

$$|J_2| \lesssim \sup_{n\in\mathcal{A}} \sum_{K_1\gtrsim M_1M_2M_3} \sum_{K_2,K_3\geqslant M_3} (1\wedge M_{min})^{1/4} \prod_{i=1}^{3} K_i^{1/2} \left|\left|f_{i,n}^{K_i}\right|\right|_{L^2}$$
$$\lesssim (1\wedge M_{min})^{1/4} \left(\frac{(1\vee M_{min})^3}{M_{min}M_{max}^2}\right)^{b_1} \prod_{i=1}^{2} ||u_{M_i}||_{\overline{F^0_{\lambda,M_i}}}$$
$$\cdot \left(\sup_{n\in\mathcal{A}} \sum_{K_1\gtrsim M_1M_2M_3} \beta_{M_1,K_1}^{b_1} K_1^{1/2} \left|\left|f_{1,n}^{K_1}\right|\right|_{L^2}\right)$$

This suffices for (3.8.1) since

$$\sup_{n\in\mathcal{A}} \sum_{K_1\gtrsim M_1M_2M_3} \beta_{M_1,K_1}^{b_1} K_1^{1/2} \left|\left|f_{1,n}^{K_1}\right|\right|_{L^2} \lesssim ||u_{M_1}||_{\overline{F^{b_1}_{\lambda,M_1}}}$$

as we only need to use (3.4.11) in this regime.

Let us now come back to (3.8.4). It remains to treat the border terms. We have

$$\sum_{n\in\mathcal{B}} \int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} f_{1,n}\star f_{2,n}\cdot f_{3,n} = \sum_{n\in\mathcal{B}} \sum_{K_1,K_2,K_3} \int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}} g_{1,n}^{K_1}\star g_{2,n}^{K_2}\cdot g_{3,n}^{K_3}$$

where $g_{i,n}^{K_i}$ is defined as

$$g_{i,n}^{K_i} := \rho_{K_i}(\tau-\omega)\mathcal{F}\left(\gamma(M_3t-n)\mathbb{1}_{[0,T]}u_{M_i}\right),\ i=1,2,3,\ K_i\geqslant 1 \tag{3.8.7}$$

Once again, we separate the different cases of corollary 3.6.9. Let us define $G_i$ the contribution of the region $A_i$ in the sum above.

Using (3.6.27), we can control the first term :

$$|G_1| \lesssim \left(M_{min}\wedge M_{min}^{-1}\right)^{1/2} M_{max}^{-1} \sup_{n\in\mathcal{B}} \sum_{(K_1,K_2,K_3)\in A_1} \prod_{i=1}^{3} K_i^{1/2} \left|\left|g_{i,n}^{K_i}\right|\right|_{L^2} \tag{3.8.8}$$

Now, we need to replace (3.8.6) by an analogous estimate on $\mathcal{B}$ :

$$\sup_{n\in\mathcal{B}} \sup_{K_i\geqslant M_3} K_i^{1/2} \left|\left|g_{i,n}^{K_i}\right|\right|_{L^2} \lesssim ||u_{M_i}||_{\overline{F^0_{\lambda,M_i}}} \tag{3.8.9}$$

Let us prove this estimate. Using the definition of $g_{i,n}^{K_i}$ (3.8.7), if we note $\widetilde{u_{M_i}} := \gamma(M_3t-n)u_{M_i}$ then we have to estimate

$$\left|\left|g_{i,n}^{K_i}\right|\right|_{L^2} = \left|\left|\rho_{K_i}(\tau-\omega)\cdot\widehat{\mathbb{1}_{[0,T]}}\star\mathcal{F}\left(\widetilde{u_{M_i}}\right)\right|\right|_{L^2}$$

We then split $\mathcal{F}(\widetilde{u_{M_i}})$ depending on its modulations :

$$\left|\left|g_{i,n}^{K_i}\right|\right|_{L^2} \leqslant \sum_{K\leqslant K_i/10} \left|\left|\rho_{K_i}(\tau-\omega)\cdot\widehat{\mathbb{1}_{[0,T]}}\star_\tau\left(\rho_K(\tau'-\omega)\mathcal{F}(\widetilde{u_{M_i}})\right)\right|\right|_{L^2}$$
$$+\sum_{K\geqslant K_i/10}\left|\left|\rho_{K_i}(\tau-\omega)\mathcal{F}_t\left\{\mathbb{1}_{[0,T]}\mathcal{F}_t^{-1}\left(\rho_K(\tau'-\omega)\mathcal{F}(\widetilde{u_{M_i}})\right)\right\}\right|\right|_{L^2} = I + II$$

To treat $I$, we use that $\left|\widehat{\mathbb{1}_{[0,T]}}(\tau-\tau')\right| \leqslant |\tau-\tau'|^{-1}\sim K_i^{-1}$ since $|\tau-\omega|\sim K_i$ and $|\tau'-\omega|\sim K\leqslant K_i/10$. Thus, from Young inequality $L^\infty\times L^1\to L^\infty$ we deduce that

$$K_i^{1/2}\cdot I \lesssim K_i\left|\left|\widehat{\mathbb{1}_{[0,T]}}\star_\tau\left(\rho_K(\tau'-\omega)\mathcal{F}(\widetilde{u_{M_i}})\right)\right|\right|_{L^2_{\xi,q}L^\infty_\tau} \lesssim ||\rho_K(\tau'-\omega)\mathcal{F}(\widetilde{u_{M_i}})||_{L^2_{\xi,q}L^1_\tau}$$

which is enough for (3.8.9) due to (3.4.12) and then (3.4.10)-(3.4.11).

To deal with $II$, we simply neglect the localization $\rho_{K_i}(\tau-\omega)$, use Plancherel identity, then neglect the localization $\mathbb{1}_{[0;T]}$ and use Plancherel identity again and that $K_i^{1/2}\lesssim K^{1/2}$ to get

$$K_i^{1/2}\cdot II \lesssim \sum_{K\geqslant K_i/10} K^{1/2}\,||\rho_K(\tau'-\omega)\mathcal{F}(\widetilde{u_{M_i}})||_{L^2_{\xi,q,\tau}} \lesssim ||\mathcal{F}(\widetilde{u_{M_i}})||_{X^0_{\lambda,M_i}}$$

This proves (3.8.9) after using again (3.4.10)-(3.4.11).

Coming back to (3.8.8) and using (3.8.9) along with $\#\mathcal{B}\leqslant 4$, we then infer

$$|G_1| \lesssim \langle\ln(M_1M_2M_3)\rangle^3(M_{min}\wedge M_{min}^{-1})^{1/2}M_{max}^{-1}\prod_{i=1}^3||u_i||_{\overline{F^0_{\lambda,M_i}}}$$

as $\displaystyle\sum_{(K_1,K_2,K_3)\in A_1} 1 \lesssim \langle\ln(M_1M_2)\rangle^3$. This is enough for (3.8.1).

Let us now turn to $G_2$. We use (3.6.28) combined with (3.8.9) to get

$$|G_2| \lesssim (1\wedge M_{min})^{1/4}M_{min}^{0+}M_{max}^{(-1)+}\sum_{(K_1,K_2,K_3)\in A_2} K_{max}^{0-}\prod_{i=1}^3\sup_{n\in\mathcal{B}}\sup_{K_i} K_i^{1/2}\left|\left|g_{i,n}^{K_i}\right|\right|_{L^2}$$
$$\lesssim M_{max}^{(-1)+}\prod_{i=1}^3||u_{M_i}||_{\overline{F^0_{\lambda,M_i}}}$$

which is sufficient as well.

Finally, we treat $G_3$, using now (3.6.29) and (3.8.9) :

$$|G_3| \lesssim (1\vee M_{min})^{(1/4)+}M_{max}^{(-5/4)+}\sum_{(K_1,K_2,K_3)\in A_3} K_{max}^{0-}\prod_{i=1}^3\sup_{n\in\mathcal{B}}\sup_{K_i}\left|\left|g_{i,n}^{K_i}\right|\right|_{L^2}$$
$$\lesssim M_{max}^{(-1)+}\prod_{i=1}^3||u_{M_i}||_{\overline{F^0_{\lambda,M_i}}}$$

which concludes the proof of lemma 3.8.1.

□

Following [IKT08, Lemma 6.1 (b)], we then use the previous estimate to control the special terms in the energy estimate 3.1.13 :

**Lemma 3.8.2**

*Let* $T\in]0;1]$, $b_1\in[0;1/8]$, $M,M_1\in 2^{\mathbb{Z}}$, *with* $M\geqslant 10(1\vee M_1)$, *and* $u\in\overline{F^{b_1}_{\lambda,M}}(T)$, $v\in\overline{F^{b_1}_{\lambda,M_1}}(T)$. *Then*

$$\left|\int_{[0,T]\times\mathbb{R}\times\mathbb{T}_\lambda} P_Mu\cdot P_M(P_{M_1}v\cdot\partial_x u)\mathrm{d}t\mathrm{d}x\mathrm{d}y\right|$$
$$\lesssim M_1\Lambda_{b_1}(M_1,M)\,||P_{M_1}v||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}\sum_{M_2\sim M}||P_{M_2}u||^2_{\overline{F^{b_1}_{\lambda,M_2}}(T)}\quad(3.8.10)$$

**Proof :**
First, we split the integral in the left-hand side of (3.8.10) into two terms

$$\int_{[0,T]\times\mathbb{R}\times\mathbb{T}_\lambda} P_Mu\cdot P_M(\partial_x uP_{M_1}v)$$
$$=\int_{[0,T]\times\mathbb{R}\times\mathbb{T}_\lambda} P_Mu\cdot P_M\partial_x u\cdot P_{M_1}v+\int_{[0,T]\times\mathbb{R}\times\mathbb{T}_\lambda} P_Mu\cdot[P_M(\partial_x uP_{M_1}v)-P_M\partial_x u\cdot P_{M_1}v]$$
$$=I+II$$

The first term is easy to control : integrating by parts and using (3.8.1), we get the bound

$$|I|=\left|\frac{1}{2}\int_{[0,T]\times\mathbb{R}\times\mathbb{T}_\lambda}(P_Mu)^2\cdot\partial_xP_{M_1}v\right|\lesssim M_1\Lambda_{b_1}(M_1,M)\,||P_Mu||^2_{\overline{F^{b_1}_{\lambda,M}}(T)}\,||P_{M_1}v||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}$$

To deal with $II$, we proceed as for the previous lemma : after choosing some extensions (still denoted $u\in\overline{F^{b_1}_{\lambda,M}}$ and $v\in\overline{F^{b_1}_{\lambda,M_1}}$) of $u$ and $v$ to $\mathbb{R}$, we split the integral in

$$II=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}^2\times\mathbb{T}_\lambda}P_Mu_n\cdot[P_M(\partial_xu_nP_{M_1}v_n)-P_M\partial_xu_n\cdot P_{M_1}v_n]$$

where we define $u_n:=\mathbb{1}_{[0,T]}\gamma(Mt-n)u$ and $v_n:=\mathbb{1}_{[0,T]}\gamma(Mt-n)v$ for a function $\gamma$ as in the previous lemma.

Using Plancherel identity, we can write $II$ as

$$II=\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}}\widehat{P_Mu_n}\cdot\int_{\mathbb{R}^2\times\lambda^{-1}\mathbb{Z}}K(\zeta,\zeta_1)\widehat{u_n}(\zeta-\zeta_1)\widehat{\partial_xP_{M_1}v_n}(\zeta_1)\mathrm{d}\zeta_1\mathrm{d}\zeta$$

where the kernel $K$ is given by

$$K(\zeta,\zeta_1)=\frac{\xi-\xi_1}{\xi_1}\left[\eta_M(\xi)-\eta_M(\xi-\xi_1)\right]\widetilde{\eta_{M_1}}(\xi_1)\sum_{M_2\sim M}\eta_{M_2}(\xi-\xi_1)$$

The last sum appears since $|\xi|\sim M$ and $|\xi_1|\sim M_1\leqslant M/10$, thus $|\xi-\xi_1|\sim M$.
Using the mean value theorem, we can bound the kernel with

$$|K(\zeta,\zeta_1)|\lesssim\left|\frac{\xi-\xi_1}{\xi_1}\right|M^{-1}|\xi_1|\widetilde{\eta_{M_1}}(\xi_1)\sum_{M_2\sim M}\eta_{M_2}(\xi-\xi_1)\lesssim\widetilde{\eta_{M_1}}(\xi_1)\sum_{M_2\sim M}\eta_{M_2}(\xi-\xi_1)\quad(3.8.11)$$

Therefore, as in [IKT08, Lemma 6.1 (b)], (3.8.10) follows after repeating the proof of (3.8.1) and using (3.8.11).

□

We finally prove (3.1.13). From now on, we fix $b_1 = 1/8$ and drop the parameter when writing the main spaces.

**Proposition 3.8.3**

*Let $T \in ]0;1]$ and $u \in \mathcal{C}([-T,T],\mathbf{E}_\lambda^\infty)$ be a solution of*

$$\begin{cases}\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y u + u\partial_x u = 0\\ u(0,x) = u_0(x)\end{cases} \tag{3.8.12}$$

*on $[-T,T]$. Then for any $\alpha \geqslant 1$,*

$$||u||_{\mathbf{B}_\lambda^\alpha(T)}^2 \lesssim ||u_0||_{\mathbf{E}_\lambda^\alpha}^2 + ||u||_{\mathbf{F}_\lambda(T)} ||u||_{\mathbf{F}_\lambda^\alpha(T)}^2 \tag{3.8.13}$$

**Proof :**

Using the definitions of $\mathbf{B}_\lambda^\alpha(T)$ (3.4.4) and $p$ (3.2.1) along with (3.4.5), it suffices to prove

$$\sum_{M_3\geqslant 1}\sup_{t_{M_3}\in[-T;T]} M_3^{2\alpha}\,||P_{M_3}u(t_{M_3})||_{L^2}^2 - M_3^{2\alpha}\,||P_{M_3}u_0||_{L^2}^2 \lesssim ||u||_{\mathbf{F}_\lambda(T)}\sum_{M_3\geqslant 1} M_3^{2\alpha}\,||P_{M_3}u||_{F^{b_1}_{\lambda,M_3}(T)}^2 \tag{3.8.14}$$

and

$$\sum_{M_3\geqslant 1}\sup_{t_{M_3}\in[-T;T]} M_3^{2(\alpha-1)}\left|\left|P_{M_3}\partial_x^{-1}\partial_y u(t_{M_3})\right|\right|_{L^2}^2 - M_3^{2(\alpha-1)}\left|\left|P_{M_3}\partial_x^{-1}\partial_y u_0\right|\right|_{L^2}^2 \lesssim ||u||_{\mathbf{F}_\lambda(T)}\sum_{M_3\geqslant 1} M_3^{2\alpha}\,||P_{M_3}u||_{F^{b_1}_{\lambda,M_3}(T)}^2 \tag{3.8.15}$$

Let us start with (3.8.14).
Applying $P_{M_3}$ to (3.8.12), multiplying by $P_{M_3}u$ and integrating, we get

$$||P_{M_3}u(t_{M_3})||_{L^2}^2 - ||P_{M_3}u_0||_{L^2}^2 = \int_0^{t_{M_3}} \frac{\mathrm{d}}{\mathrm{d}t}\,||P_{M_3}u(t)||_{L^2}^2\,\mathrm{d}t \lesssim \left|\int_0^{t_{M_3}}\int_{\mathbb{R}\times\mathbb{T}_\lambda} P_{M_3}u\cdot P_{M_3}(u\partial_x u)\mathrm{d}t'\mathrm{d}x\mathrm{d}y\right| \tag{3.8.16}$$

since $\partial_x^3$ and $\partial_x^{-1}\partial_y^2$ are skew-adjoint.
We separate the right-hand side of (3.8.16) in

$$\sum_{M_1\leqslant M_3/10}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda} P_{M_3}u\cdot P_{M_3}\left(P_{M_1}u\cdot\partial_x u\right)\mathrm{d}t\mathrm{d}x\mathrm{d}y \tag{3.8.17}$$

$$+\sum_{M_1\gtrsim M_3}\sum_{M_2>0}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda} (P_{M_3})^2u\cdot P_{M_1}u\cdot\partial_x P_{M_2}u\mathrm{d}t\mathrm{d}x\mathrm{d}y \tag{3.8.18}$$

Using (3.8.10) and Cauchy-Schwarz inequality in $M_1$, the first term (3.8.17) is estimated by

$$(3.8.17) \lesssim \sum_{M_1 \leqslant M_3/10} M_1 \Lambda_{b_1}(M_1, M_3) \, ||P_{M_1} u||_{\overline{F^{b_1}_{\lambda,M_1}}(T)} \sum_{M_2 \sim M_3} ||P_{M_2} u||^2_{\overline{F^{b_1}_{\lambda,M_2}}(T)}$$
$$\lesssim ||u||_{\mathbf{F}_\lambda(T)} \sum_{M_2 \sim M} ||P_{M_2} u||^2_{\overline{F^{b_1}_{\lambda,M_2}}(T)}$$

since

$$\left[ \sum_{0 < M_1 \leqslant M_3/10} (1 \vee M_1)^{-2} M_1^2 \Lambda_{b_1}(M_1, M_3)^2 \right]^{1/2} \lesssim 1$$

Thus

$$\sum_{M_3 \geqslant 1} M_3^{2\alpha} \cdot (3.8.17) \lesssim ||u||_{\mathbf{F}_\lambda(T)} \, ||u||^2_{\mathbf{F}^\alpha_\lambda(T)}$$

To treat (3.8.18), we use (3.8.1) and then we separate the sum on $M_2$ depending on whether $M_1 \sim M_3 \gtrsim M_2$ or $M_1 \sim M_2 \gtrsim M_3$ :

$$(3.8.18) \lesssim \sum_{M_1 \sim M_3} \sum_{M_2 \lesssim M_3} M_2 \Lambda_{b_1}(M_2, M_3) \prod_{i=1}^{3} ||P_{M_i} u||_{\overline{F^{b_1}_{\lambda,M_i}}(T)}$$
$$+ \sum_{M_1 \gtrsim M_3} \sum_{M_2 \sim M_1} M_2 \Lambda_{b_1}(M_3, M_2) \prod_{i=1}^{3} ||P_{M_i} u||_{\overline{F^{b_1}_{\lambda,M_i}}(T)} = I + II$$

Applying Cauchy-Schwarz inequality in $M_2$ we get the bounds

$$I \lesssim ||P_{M_3} u||^2_{\overline{F^{b_1}_{\lambda,M_3}}(T)} \, ||u||_{\mathbf{F}_\lambda(T)}$$

and

$$II \lesssim \sum_{M_1 \gtrsim M_3} M_1 \Lambda_{b_1}(M_3, M_1) \, ||P_{M_3} u||_{\overline{F^{b_1}_{\lambda,M_3}}(T)} \, ||P_{M_1} u||^2_{\overline{F^{b_1}_{\lambda,M_1}}(T)}$$

Summing on $M_3$ and using Cauchy-Schwarz inequality in $M_3$ and $M_1$ for $II$, we finally get

$$\sum_{M_3 \geqslant 1} M_3^{2\alpha} \cdot (3.8.18) \lesssim ||u||_{\mathbf{F}_\lambda(T)} \, ||u||^2_{\mathbf{F}^\alpha_\lambda(T)}$$
$$+ \sum_{M_3 \geqslant 1} \sum_{M_1 \gtrsim M_3} M_3^\alpha \Lambda_{b_1}(M_3, M_1) M_1^{1+\alpha} \, ||P_{M_3} u||_{\overline{F^{b_1}_{\lambda,M_3}}(T)} \, ||P_{M_1} u||^2_{\overline{F^{b_1}_{\lambda,M_1}}(T)}$$
$$\lesssim ||u||_{\mathbf{F}_\lambda(T)} \, ||u||^2_{\mathbf{F}^\alpha_\lambda(T)}$$

Now we turn to the proof of (3.8.15).
This time, we apply $P_{M_3} \partial_x^{-1} \partial_y$ to (3.1.6), we multiply by $P_{M_3} \partial_x^{-1} \partial_y u$ and we integrate to get

$$\left| \big|\big| P_{M_3} \partial_x^{-1} \partial_y u(t_{M_3}) \big|\big|^2_{L^2} - \big|\big| P_{M_3} \partial_x^{-1} \partial_y u_0 \big|\big|^2_{L^2} \right|$$
$$\lesssim \left| \int_{[0, t_{M_3}] \times \mathbb{R} \times \mathbb{T}_\lambda} P_{M_3} \partial_x^{-1} \partial_y u \cdot P_{M_3} \partial_x^{-1} \partial_y (u \partial_x u) \mathrm{d}t \mathrm{d}x \mathrm{d}y \right| \quad (3.8.19)$$

using again the skew-adjointness of $\partial_x^3$ and $\partial_x^{-1}\partial_y^2$.
The right-hand side of (3.8.19) is similarly split up into

$$\sum_{M_1\leqslant M_3/10}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda} P_{M_3}\partial_x^{-1}\partial_y u\cdot P_{M_3}(P_{M_1}u\cdot\partial_y u)\mathrm{d}t\mathrm{d}x\mathrm{d}y \tag{3.8.20}$$

$$+\sum_{M_1\gtrsim M_3}\sum_{M_2}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda}(P_{M_3})^2\partial_x^{-1}\partial_y u\cdot P_{M_1}u\cdot\partial_y P_{M_2}u\mathrm{d}t\mathrm{d}x\mathrm{d}y \tag{3.8.21}$$

Writing $v:=\partial_x^{-1}\partial_y u$, using (3.8.10) and Cauchy-Schwarz inequality in $M_1$, we obtain

$$\begin{aligned}(3.8.20)\lesssim\sum_{M_1\leqslant M_3/10} M_1\Lambda_{b_1}(M_1,M_3)\,||P_{M_1}u||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}\sum_{M_2\sim M_3}||v_{M_2}||^2_{\overline{F^{b_1}_{\lambda,M_2}}(T)}\\ \lesssim ||u||_{\mathbf{F}_\lambda(T)}\sum_{M_2\sim M_3}\left|\left|P_{M_2}\partial_x^{-1}\partial_y u\right|\right|^2_{\overline{F^{b_1}_{\lambda,M_2}}(T)}\end{aligned}$$

which is enough for (3.8.15) after summing on $M_3$.

As for (3.8.21), we separate again the sum on $M_2$ :

$$\begin{aligned}(3.8.21)=I+II=\sum_{M_1\sim M_3}\sum_{M_2\lesssim M_3}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda}(P_{M_3})^2 v\cdot P_{M_1}u\cdot\partial_x v_{M_2}\mathrm{d}t\mathrm{d}x\mathrm{d}y\\ +\sum_{M_1\gtrsim M_3}\sum_{M_2\sim M_1}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda}(P_{M_3})^2 v\cdot P_{M_1}u\cdot\partial_x P_{M_2}v\mathrm{d}t\mathrm{d}x\mathrm{d}y\end{aligned}$$

For the first term, we use again (3.8.1) which gives

$$I\lesssim\sum_{M_1\sim M_3}\sum_{M_2\lesssim M_3}M_2\Lambda_{b_1}(M_2,M_3)\,||P_{M_3}v||_{\overline{F^{b_1}_{\lambda,M_3}}(T)}\,||P_{M_1}u||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}\,||P_{M_2}v||_{\overline{F^{b_1}_{\lambda,M_2}}(T)}$$

We first sum on $M_2$ by using Cauchy-Schwarz inequality to get the bound

$$\sum_{M_1\sim M_3}M_3\,||P_{M_3}v||_{\overline{F^{b_1}_{\lambda,M_3}}(T)}\,||P_{M_1}u||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}\,||u||_{\mathbf{F}_\lambda(T)}$$

and then we can sum on $M_3$ using Cauchy-Schwarz inequality again to get (3.8.15) for this term.

For the second term, we apply also (3.8.1), then we first sum on $M_3$ using Cauchy-Schwarz inequality and $M_3^{2(\alpha-1)}\lesssim M_1^{2(\alpha-1)}$ in this regime, and finally sum on $M_1$ using again Cauchy-Schwarz inequality to get (3.8.15).

□

In the same spirit, following [IKT08] we have for the difference equation

**Proposition 3.8.4**

*Let $T\in]0,1[$ and $u,v\in\mathbf{F}_\lambda(T)$ satisfying*

$$\begin{cases}\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y u+\partial_x(uv)=0\\ u(0,x)=u_0(x)\end{cases} \tag{3.8.22}$$

*on* $[-T,T]\times\mathbb{R}\times\mathbb{T}_\lambda$. *Then*

$$||u||^2_{\mathbf{B}_\lambda(T)} \lesssim ||u_0||^2_{L^2_\lambda} + ||v||_{\mathbf{F}_\lambda(T)}\,||u||^2_{\overline{\mathbf{F}_\lambda}(T)} \tag{3.8.23}$$

and to deal with the equation satisfied by $P_{High}\partial_x(u_1-u_2)$ we need

**Proposition 3.8.5**

*Let* $T\in]0;1]$ *and* $u\in\overline{\mathbf{F}_\lambda}(T)$ *with* $u=P_{High}u$. *Moreover, let* $v\in\mathbf{F}_\lambda(T)$, $w_i\in\mathbf{F}_\lambda(T)$, $i=1,2,3$, *and* $w_i'\in\overline{\mathbf{F}_\lambda}(T)$, $i=1,2,3$ *and finally* $h\in\overline{\mathbf{F}_\lambda}(T)$ *with* $h=P_{\leqslant 1}h$. *Assume that* $u$ *satisfies*

$$\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u = P_{High}(v\partial_x u)+\sum_{i=1}^3 P_{High}(w_iw_i')+P_{High}h \tag{3.8.24}$$

*on* $[-T;T]\times\mathbb{R}\times\mathbb{T}_\lambda$. *Then*

$$||u||^2_{\overline{\mathbf{B}_\lambda}(T)} \lesssim ||u_0||^2_{L^2_\lambda} + ||v||_{\mathbf{F}_\lambda(T)}\,||u||^2_{\overline{\mathbf{F}_\lambda}(T)} + ||u||_{\overline{\mathbf{F}_\lambda}(T)}\sum_{i=1}^3 ||w_i||_{\overline{\mathbf{F}_\lambda}(T)}\,||w_i'||_{\overline{\mathbf{F}_\lambda}(T)} \tag{3.8.25}$$

**Proof :**
(3.8.23) follows from (3.8.25) after splitting up $u$ into $P_{Low}u$ and $P_{High}u$ and observing that $P_{High}u$ satisfies an equation of type (3.8.24).

To prove (3.8.25), we follow the proof of proposition 3.8.3. Using the definitions of $\overline{\mathbf{B}_\lambda}(T)$ (3.4.6), it suffices to prove

$$\sum_{M_3>1}\sup_{t_{M_3}\in[-T;T]} ||P_{M_3}u(t_{M_3})||^2_{L^2}-||P_{M_3}u_0||^2_{L^2} \lesssim ||v||_{\mathbf{F}_\lambda(T)}\,||u||^2_{\overline{\mathbf{F}_\lambda}(T)} + ||u||_{\overline{\mathbf{F}_\lambda}(T)}\sum_{i=1}^3 ||w_i||_{\overline{\mathbf{F}_\lambda}(T)}\,||w_i'||_{\overline{\mathbf{F}_\lambda}(T)} \tag{3.8.26}$$

Take $M_3>1$. Applying $P_{M_3}$ to (3.8.24), multiplying by $P_{M_3}u$ and integrating, we get

$$\begin{aligned}||P_{M_3}u(t_{M_3})||^2_{L^2}-||P_{M_3}u_0||^2_{L^2} &= \int_0^{t_{M_3}}\frac{\mathrm{d}}{\mathrm{d}t}||P_{M_3}u(t)||^2_{L^2}\,\mathrm{d}t\\ &\lesssim \left|\int_0^{t_{M_3}}\int_{\mathbb{R}\times\mathbb{T}_\lambda} P_{M_3}u\cdot P_{M_3}P_{High}(u\partial_x v)\mathrm{d}t'\mathrm{d}x\mathrm{d}y\right|\\ &\quad+\sum_{i=1}^3\left|\int_0^{t_{M_3}}\int_{\mathbb{R}\times\mathbb{T}_\lambda} P_{M_3}u\cdot P_{M_3}P_{High}(w_iw_i')\mathrm{d}t'\mathrm{d}x\mathrm{d}y\right|\end{aligned} \tag{3.8.27}$$

since $\partial_x^3$ and $\partial_x^{-1}\partial_y^2$ are skew-adjoint. The term in $h$ vanishes after applying $P_{M_3}$, due to its frequency localization.

To treat the first term in the right-hand side of (3.8.27) we split it up in

$$\sum_{M_1\leqslant M_3/10}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda} P_{M_3}u\cdot P_{M_3}\left(P_{M_1}v\cdot\partial_x u\right)\mathrm{d}t\mathrm{d}x\mathrm{d}y \tag{3.8.28}$$

$$+\sum_{M_1\gtrsim M_3}\sum_{M_2}\int_{[0,t_{M_3}]\times\mathbb{R}\times\mathbb{T}_\lambda}(P_{M_3})^2u\cdot P_{M_1}v\cdot\partial_x P_{M_2}u\mathrm{d}t\mathrm{d}x\mathrm{d}y \tag{3.8.29}$$

The first term (3.8.28) is estimated similarly to (3.8.20) with $\alpha=1$ and exchanging the roles of $u$ and $v$, whereas for (3.8.29) we proceed as for (3.8.21).

To treat the second term in the right-hand side of (3.8.27), we perform a dyadic decomposition of $w_i$ and $w_i'$. By symmetry we can assume $M_1\leqslant M_2$, thus either $M_1\lesssim M_2\sim M_3$ or $M_3\lesssim M_1\sim M_2$. Then we apply (3.8.1) to bound the sum on $M_3$ by

$$\sum_{M_3\geqslant 1}\sum_{M_2\sim M_3}\sum_{M_1\lesssim M_2}\Lambda_{b_1}(M_1,M_2)$$
$$\cdot\,||P_{M_3}u||_{\overline{F^{b_1}_{\lambda,M_3}}(T)}\,||P_{M_1}w_i||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}\,||P_{M_2}w_i'||_{\overline{F^{b_1}_{\lambda,M_2}}(T)}$$
$$+\sum_{M_2\geqslant 1}\sum_{M_1\sim M_2}\sum_{1\leqslant M_3\lesssim M_2}\Lambda_{b_1}(M_3,M_2)$$
$$\cdot\,||P_{M_3}u||_{\overline{F^{b_1}_{\lambda,M_3}}(T)}\,||P_{M_1}w_i||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}\,||P_{M_2}w_i'||_{\overline{F^{b_1}_{\lambda,M_2}}(T)}$$

For the second term, we can just use Cauchy-Schwarz inequality in $M_3$ and $M_2$ since $M_1\sim M_2\gtrsim M_3\geqslant 1$. For the first term, we use that

$$\sum_{0<M_1\lesssim M_2}\Lambda_{b_1}(M_1,M_2)\,||P_{M_1}w_i||_{\overline{F^{b_1}_{\lambda,M_1}}(T)}\lesssim ||w_i||_{\overline{\mathbf{F}_\lambda}(T)}$$

Note that this is the only step where we need (3.6.8) to avoid a logarithmic divergence when summing on very low frequencies, thus we do not need the extra decay for low frequency as in [IKT08].

Thus we finally obtain

$$\sum_{M\geqslant 1}(3.8.29)\lesssim\sum_{i=1}^{3}||u||_{\overline{\mathbf{F}_\lambda(T)}}\,||w_i||_{\overline{\mathbf{F}_\lambda(T)}}\,||w_i'||_{\overline{\mathbf{F}_\lambda(T)}}$$

which concludes the proof of (3.8.25).

□

## 3.9 Proof of Theorem 3.1.1

We finally turn to the proof of our main result. We follow the scheme of [KP15, Section 6].

We begin by recalling a local well-posedness result for smooth data :

**Proposition 3.9.1**

*Assume $u_0\in\mathbf{E}_\lambda^\infty$. Then there exists $T_\lambda\in]0;1]$ and a unique solution $u\in\mathcal{C}([-T_\lambda;T_\lambda],\mathbf{E}_\lambda^\infty)$*

*of (3.1.6) on $[-T_\lambda; T_\lambda] \times \mathbb{R} \times \mathbb{T}_\lambda$. Moreover, $T_\lambda = T(||u_0||_{\mathbf{E}^3_\lambda})$ can be chosen as a nonincreasing function of $||u_0||_{\mathbf{E}^3_\lambda}$.*

**Proof :**
This is a straightforward adaptation of [IN98] to the case of partially periodic data. Indeed, proposition 3.9.1 follows from the standard energy estimate (see for example [Ken04, Lemma 1.3])

$$||u||_{L^\infty_T \mathbf{E}^\alpha_\lambda} \leqslant C_\alpha \, ||u_0||_{\mathbf{E}^\alpha_\lambda} \exp\left(\widetilde{C}_\alpha \, ||\partial_x u||_{L^1_T L^\infty_{xy}}\right) \tag{3.9.1}$$

along with the Sobolev embedding

$$||\partial_x u||_{L^1_T L^\infty_{xy}} \lesssim T \, ||u||_{L^\infty_T \mathbf{E}^3_\lambda}$$

□

### 3.9.1 A priori estimates for smooth solutions

In this subsection we improve the control on the previous solutions.

**Proposition 3.9.2**

*There exists $\epsilon_0 \in ]0;1]$ such that for $u_0 \in \mathbf{E}^\infty_\lambda$ with*

$$||u_0||_{\mathbf{E}_\lambda} \leqslant \epsilon_0 \tag{3.9.2}$$

*then there exists a unique solution $u$ to (3.1.6) in $\mathcal{C}([-1;1], \mathbf{E}^\infty_\lambda)$, and it satisfies for $\alpha = 1, 2, 3$,*

$$||u||_{\mathbf{F}^\alpha_\lambda(1)} \leqslant C_\alpha \, ||u_0||_{\mathbf{E}^\alpha_\lambda} \tag{3.9.3}$$

*where $C_\alpha > 0$ is a constant independent of $\lambda$.*

**Proof :**
Let $T = T\left(||u_0||_{\mathbf{E}^3_\lambda}\right) \in ]0;1]$ and $u \in \mathcal{C}([-T;T], \mathbf{E}^\infty_\lambda)$ be the solution to (3.1.6) given by proposition 3.9.1. Then, for $T' \in [0;T]$, we define

$$\mathcal{X}_{\lambda,\alpha}(T') := ||u||_{\mathbf{B}^\alpha_\lambda(T')} + ||u\partial_x u||_{\mathbf{N}^\alpha_\lambda(T')} \tag{3.9.4}$$

Recalling (3.5.2)-(3.7.15)-(3.8.13) for $\alpha \in \mathbb{N}^*$, we get

$$\begin{cases} ||u||_{\mathbf{F}^\alpha_\lambda(T)} \lesssim ||u||_{\mathbf{B}^\alpha_\lambda(T)} + ||f||_{\mathbf{N}^\alpha_\lambda(T)} \\ ||\partial_x(uv)||_{\mathbf{N}^\alpha_\lambda(T)} \lesssim ||u||_{\mathbf{F}^\alpha_\lambda(T)} \, ||v||_{\mathbf{F}_\lambda(T)} + ||u||_{\mathbf{F}_\lambda(T)} \, ||v||_{\mathbf{F}^\alpha_\lambda(T)} \\ ||u||^2_{\mathbf{B}^\alpha_\lambda(T)} \lesssim ||u_0||^2_{\mathbf{E}^\alpha_\lambda} + ||u||_{\mathbf{F}_\lambda(T)} \, ||u||^2_{\mathbf{F}^\alpha_\lambda(T)} \end{cases} \tag{3.9.5}$$

Thus, combining those estimates first with $\alpha = 1$, we deduce that

$$\mathcal{X}_{\lambda,1}(T')^2 \leqslant c_1 \, ||u_0||^2_{\mathbf{E}_\lambda} + c_2 \left(\mathcal{X}_{\lambda,1}(T')^3 + \mathcal{X}_{\lambda,1}(T')^4\right) \tag{3.9.6}$$

Let us remind here that the constants appearing in (3.5.2)-(3.7.15)-(3.8.13) do *not* depend on $\lambda \geqslant 1$, so neither does (3.9.6). Thus, using lemma 3.9.3 below and a continuity argument, we get that there exists $T_0 = T_0(\epsilon_0) \in ]0;1]$ such that $\mathcal{X}_{\lambda,1}(T) \leqslant 2c_0\epsilon_0$ for $T \in [0;T_0]$. Thus, if we choose $\epsilon_0$ small enough such that

$$2c_2c_0\epsilon_0 + 4c_2c_0^2\epsilon_0^2 < \frac{1}{2}$$

then

$$\mathcal{X}_{\lambda,1}(T) \lesssim ||u_0||_{\mathbf{E}_\lambda}$$

for $T \in [0;T_0]$.

(3.9.3) for $\alpha = 1$ then follows from (3.5.2).

Next, substituting the estimate obtained above in (3.9.5), we infer that for $\alpha = 2, 3$

$$\mathcal{X}_{\lambda,\alpha}(T)^2 \leqslant c_\alpha \, ||u_0||^2_{\mathbf{E}^\alpha_\lambda} + \widetilde{c_\alpha}\epsilon_0\mathcal{X}_{\lambda,\alpha}(T)^2$$

which in turn, up to chosing $\epsilon_0$ even smaller such that $\widetilde{c_\alpha}\epsilon_0 < 1/2$, gives (3.9.3) for $\alpha = 2, 3$.

To reach $T = 1$, we just have to use (3.9.3) with $\alpha = 3$ along with (3.4.16), and then extend the lifespan of $u$ by using proposition 3.9.1 a finite number of times.

□

Therefore it remains to prove the following lemma :

**Lemma 3.9.3**

*Let $T \in ]0;1]$ and $u \in \mathcal{C}([-T;T], \mathbf{E}^\infty_\lambda)$. Then $\mathcal{X}_{\lambda,1} : [0;T] \to \mathbb{R}$, defined in (3.9.4), is continuous and nondecreasing, and furthermore*

$$\lim_{T'\to 0} \mathcal{X}_{\lambda,1}(T') \leqslant c_0 \, ||u_0||_{\mathbf{E}_\lambda}$$

*where $c_0 > 0$ is a constant independent of $\lambda$.*

**Proof :**

From the definition of $\mathbf{B}_\lambda(T)$ (3.4.4) it is clear that for $u \in \mathcal{C}([-T;T], \mathbf{E}^\infty_\lambda)$,
$T' \mapsto ||u||_{\mathbf{B}_\lambda(T')}$ is nondecreasing and continuous and satisfies

$$\lim_{T'\to 0} ||u||_{\mathbf{B}_\lambda(T')} \lesssim ||u_0||_{\mathbf{E}_\lambda}$$

where the constant only depends on the choice of the dyadic partition of unity.

Thus it remains to prove that for all $v \in \mathcal{C}([-T;T], \mathbf{E}^\infty_\lambda)$, $T' \mapsto ||v||_{\mathbf{N}_\lambda(T')}$ is increasing and continuous on $[0;T]$ and satisfies

$$\lim_{T'\to 0} ||v||_{\mathbf{N}_\lambda(T')} = 0 \tag{3.9.7}$$

The proof is the same as in [KP15, Lemma 6.3] or [GO16, Lemma 8.1] : first, for $M > 0$ and $T' \in [0;T]$, take an extension $v_M$ of $P_M v$ outside of $[-T;T]$, then using the definition of $N^{b_1}_{\lambda,M}$ we get

$$||P_M v||_{N^{b_1}_{\lambda,M}(T')} \lesssim ||\chi_{T'}(t)v_M||_{N^{b_1}_{\lambda,M}} \lesssim ||p \cdot \mathcal{F}\left\{\chi_{T'}(t)v_M\right\}||_{L^2}$$

Using the Littlewood-Paley theorem, we obtain the bound

$$||v||_{\mathbf{N}_\lambda(T')} = \left(\sum_{M>0}(1\vee M)^2\,||P_M v||^2_{N^{b_1}_{\lambda,M}(T')}\right)^{1/2}$$
$$\lesssim \left(\sum_{M>0}(1\vee M)^2\,||p\cdot\mathcal{F}\{\chi_{T'}(t)v_M\}||^2_{L^2}\right)^{1/2}$$
$$\lesssim ||\chi_{T'}v||_{L^2_T\mathbf{E}_\lambda}\lesssim (T')^{1/2}\,||v||_{L^\infty_T\mathbf{E}_\lambda} \quad (3.9.8)$$

This proves (3.9.7) and the continuity at $T'=0$. The nondecreasing property follows from the definition of $||\cdot||_{Y(T')}$ (3.4.1). It remains to prove the continuity in $T_0\in]0;T]$.

Let $\epsilon>0$. If we define for $u_0\in L^2(\mathbb{R}\times\mathbb{T})$ and $L>0$,

$$\mathbb{P}_L u_0 := \mathcal{F}^{-1}\left\{\chi_L(\omega(\xi,q))\widehat{u_0}\right\}$$

then by monotone convergence theorem we can take $L$ large enough such that

$$||(\mathrm{Id}-\mathbb{P}_L)v||_{\mathbf{N}_\lambda(T_0)}<\epsilon$$

Then it suffices to show that there exists $\delta_0>0$ such that for $r\in[1-\delta_0;1+\delta_0]$,

$$\Big|\,||v_L||_{\mathbf{N}_\lambda(T_0)}-||v_L||_{\mathbf{N}_\lambda(rT_0)}\Big|<\epsilon$$

Thus we may assume $v=\mathbb{P}_L v$ in the sequel. In particular, $P_M v=0$ if $M^3\gtrsim L$.

As in [IKT08], we define for $r$ close to 1 the scaling operator

$$D_r(v)(t,x,y):=v(t/r,x,y)$$

Proceeding as in (3.9.8), we have

$$\big|\big|v-D_{T'/T_0}(v)\big|\big|_{\mathbf{N}_\lambda(T')}\lesssim (T')^{1/2}\,\big|\big|v-D_{T'/T_0}(v)\big|\big|_{L^\infty_T\mathbf{E}_\lambda}\underset{T'\to T_0}{\longrightarrow}0$$

where we use that $v\in\mathcal{C}([-T;T],\mathbf{E}_\lambda)$ to get the convergence.

Consequently, we are left with proving

$$||v||_{\mathbf{N}_\lambda(T_0)}\leqslant \liminf_{r\to1}||D_r(v)||_{\mathbf{N}_\lambda(rT_0)} \quad (3.9.9)$$

and

$$\limsup_{r\to1}||D_r(v)||_{\mathbf{N}_\lambda(rT_0)}\leqslant ||v||_{\mathbf{N}_\lambda(T_0)} \quad (3.9.10)$$

Let us begin with (3.9.9). Fixing $\widetilde{\epsilon}>0$ and $r\in[1/2;2]$, for any $M\in 2^{\mathbb{Z}}$, $M^3\lesssim L$, we can choose an extension $v_{M,r}$ of $P_M D_r(v)$ satisfying $v_{M,r}\equiv P_M D_r(v)$ on $[-rT_0;rT_0]$ and

$$||v_{M,r}||_{N^{b_1}_{\lambda,M}}\leqslant ||P_M D_r(v)||_{N^{b_1}_{\lambda,M}(rT_0)}+\widetilde{\epsilon}$$

Since $D_{1/r}(v_{M,r})\equiv P_M v$ on $[-T_0;T_0]$, it defines an extension of $P_M v$ and thus

$$||v||_{\mathbf{N}_\lambda(T_0)}\leqslant\left(\sum_{M\lesssim L^{1/3}}(1\vee M)^2\,\big|\big|D_{1/r}(v_{M,r})\big|\big|^2_{N^{b_1}_{\lambda,M}}\right)^{1/2}$$

Finally, it remains to prove that

$$\left|\left|D_{1/r}(v_{M,r})\right|\right|_{N^{b_1}_{\lambda,M}} \leqslant \psi(r)\left|\left|v_{M,r}\right|\right|_{N^{b_1}_{\lambda,M}} \tag{3.9.11}$$

to get (3.9.9), where $\psi$ is a continuous function defined on a neighborhood of $r=1$ and satisfying $\lim_{r\to 1}\psi(r)=1$.

From the definition of $N^{b_1}_{\lambda,M}$, we have

$$\left|\left|D_{1/r}(v_{M,r})\right|\right|_{N^{b_1}_{\lambda,M}} = \sup_{t_M\in\mathbb{R}}\left|\left|(\tau-\omega+i(1\vee M))^{-1}p\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(\cdot - t_M)D_{1/r}(v_{M,r})\right\}\right|\right|_{X^{b_1}_{\lambda,M}}$$

and a computaton gives

$$\chi_{(1\vee M)^{-1}}(\cdot - t_M)D_{1/r}(v_{M,r}) = D_{1/r}\left(\chi_{r(1\vee M)^{-1}}(\cdot - rt_M)v_{M,r}\right)$$

so that

$$\mathcal{F}\left\{\chi_{(1\vee M)^{-1}}(\cdot - t_M)D_{1/r}(v_{M,r})\right\} = r^{-1}D_r\left(\mathcal{F}\left\{\chi_{r(1\vee M)^{-1}}(\cdot - rt_M)v_{M,r}\right\}\right)$$

Thus, using the definition of $X^{b_1}_{\lambda,M}$, the left-hand side of (3.9.11) equals

$$r^{-1/2}\sup_{\widetilde{t_M}\in\mathbb{R}}\sum_{K\geqslant 1}K^{1/2}\beta^{b_1}_{M,K} \left|\left|(r\tau-\omega+i(1\vee M))^{-1}\rho_K(r\tau-\omega)p\mathcal{F}\left\{\chi_{r(1\vee M)^{-1}}(\cdot - \widetilde{t_M})v_{M,r}\right\}\right|\right|_{L^2}$$

Now, for $r\sim 1$, we observe that for $K\geqslant 10^{10}L$, we have $|\tau|\sim|\tau-\omega|\sim|r\tau-\omega|\sim K$, whereas for $K\lesssim L$ we have $|\tau|$, $|\tau-\omega|$ and $|r\tau-\omega|\lesssim L$.

Thus,

$$\left|\frac{1}{(r\tau-\omega)^2+(1\vee M)^2}-\frac{1}{(\tau-\omega)^2+(1\vee M)^2}\right| \lesssim |1-r|(1\vee L)^2\cdot\frac{1}{(\tau-\omega)^2+(1\vee M)^2} \tag{3.9.12}$$

and the use of the mean value theorem provides

$$|\rho_K(r\tau-\omega)-\rho_K(\tau-\omega)| \lesssim |1-r|\begin{cases}\displaystyle\sum_{K'\sim K}\rho_{K'}(\tau-\omega) \text{ if } K\geqslant 10^{10}L\\ \displaystyle K^{-1}L\sum_{K'\lesssim L}\rho_{K'}(\tau-\omega) \text{ if } K\lesssim L\end{cases} \tag{3.9.13}$$

Combining all the estimates above, we get the bound

$$\left|\left|D_{1/r}(v_{M,r})\right|\right|_{N^{b_1}_{\lambda,M}} \leqslant \widetilde{\psi}(r)\sup_{\widetilde{t_M}}\left|\left|(\tau-\omega+i(1\vee M))^{-1}p\mathcal{F}\left\{\chi_{r(1\vee M)^{-1}}(\cdot - \widetilde{t_M})v_{M,r}\right\}\right|\right|_{X^{b_1}_{\lambda,M}} \tag{3.9.14}$$

where $\widetilde{\psi}(r) = r^{-1/2}\left(1 + C(1 \vee L)^2|r-1|\right)^{3/2} \underset{r\to 1}{\longrightarrow} 1$.

It remains to treat the time localization term : using the fundamental theorem of calculus, we have

$$F(t-\widetilde{t_M}) := \chi_{r(1\vee M)^{-1}}(t-\widetilde{t_M}) - \chi_{(1\vee M)^{-1}}(t-\widetilde{t_M}) = \int_1^{r^{-1}} s^{-1}\varphi(s(1\vee M)(t-\widetilde{t_M}))\mathrm{d}s$$

with $\varphi(t) := t\chi'(t)$. In particular, for $r \in [1/2;2]$, from the support property of $\chi$, the support of $F(\cdot - \widetilde{t_M})$ is included in $[\widetilde{t_M} - 4(1\vee M); \widetilde{t_M} + 4(1\vee M)]$, thus we can represent

$$F(t-\widetilde{t_M}) = F(t-\widetilde{t_M}) \sum_{|\ell|\leqslant 4} \gamma((1\vee M)(t-\widetilde{t_M}-\ell)\chi_{(1\vee M)^{-1}}(t-\widetilde{t_M}-\ell(1\vee M)^{-1})$$

where $\gamma$ is a smooth partition of unity with $\mathrm{supp}\gamma \subset [-1;1]$ satisfying $\forall x \in \mathbb{R}, \sum_{\ell\in\mathbb{Z}} \gamma(x-\ell) = 1$.

Now, using Minkowski's integral inequality to deal with the integral in $s$, the right-hand-side of (3.9.14) is less than

$$\widetilde{\psi}(r)\left(||v_{M,r}||_{N^{b_1}_{\lambda,M}} + \int_{I(r)} s^{-1} \sup_{\widetilde{t_M}} \sum_{|\ell|\leqslant 4} \Big|\Big|(\tau-\omega+i(1\vee M))^{-1}p\mathcal{F}\Big\{\varphi(s(1\vee M)(t-\widetilde{t_M})\right.$$

$$\left.\cdot\gamma_{(1\vee M)^{-1}}(t-\widetilde{t_M}-(1\vee M)^{-1}\ell)\chi_{(1\vee M)^{-1}}(t-\widetilde{t_M}-(1\vee M)^{-1}\ell)v_{M,r}\Big\}\Big|\Big|_{X^{b_1}_{\lambda,M}} \mathrm{d}s\right)$$

with $I(r) = [1;r^{-1}]$ if $r \in [1/2;1]$ and $I(r) = [r^{-1};1]$ if $r \in [1;2]$.

Since $\varphi(t) = t\chi'(t)$ and $\gamma$ are smooth, twice the use of (3.4.10) and (3.4.11) (with $K_0 = s(1\vee M)$ and $K_0 = (1\vee M)$ respectively) provides the final bound

$$\big|\big|D_{1/r}(v_{M,r})\big|\big|_{N^{b_1}_{\lambda,M}} \leqslant \widetilde{\psi}(r)\,(1 + C\,|\ln(r)|)\,||v_{M,r}||_{N^{b_1}_{\lambda,M}} \tag{3.9.15}$$

(here we used that the implicit constant in (3.4.10) and (3.4.11) are independent of $s$). This concludes the proof of (3.9.9).

To prove (3.9.10), as before we may assume $v = \mathbb{P}_L v$. Given $\widetilde{\epsilon} > 0$ for any $M > 0$ we take an extension $v_M$ of $P_M v$ outside of $[-T_0;T_0]$ and satisfying $||v_M||_{N^{b_1}_{\lambda,M}} \leqslant ||P_M v||_{N^{b_1}_{\lambda,M}(T_0)} + \widetilde{\epsilon}$. Then for $r \in [-1/2;2]$, $D_r(v_M)$ defines an extension of $P_M D_r(v)$ outside of $[-rT_0;rT_0]$. Then, since in the proof of (3.9.15) we did not use the dependence in $r$ of $v_{M,r}$, the same estimate actually holds for $v_M$, and thus

$$||D_r(v_M)||_{N^{b_1}_{\lambda,M}} \leqslant \psi(1/r)\,||v_M||_{N^{b_1}_{\lambda,M}}$$

which is enough for (3.9.10) and thus concludes the proof of the lemma.

□

### 3.9.2 Global well-posedness for smooth data

In view of the previous proposition, theorem 3.1.1 (a) follows from the conservation of the energy.

Indeed, take $u_0 \in \mathbf{E}^\infty_\lambda$ satisfying

$$||u_0||_{\mathbf{E}_\lambda} \leqslant \epsilon_1 \leqslant \epsilon_0 \tag{3.9.16}$$

and let $T^* := \sup\{T \geqslant 1,\ ||u(T)||_{\mathbf{E}_\lambda} < +\infty\}$ where $u$ is the unique maximal solution of (3.1.6) given by proposition 3.9.2. Then, using the anisotropic Sobolev estimate (see [Tom96, Lemma 2.5])

$$\int_{\mathbb{R}\times\mathbb{T}_\lambda} u_0(x,y)^3 \mathrm{d}x\mathrm{d}y \leqslant 2\,||u_0||_{L^2}^{3/2}\,||\partial_x u_0||_{L^2}\,\left|\left|\partial_x^{-1}\partial_y u_0\right|\right|_{L^2}^{1/2} \tag{3.9.17}$$

we have for $T < T^*$

$$\begin{aligned} ||u(T)||_{\mathbf{E}_\lambda}^2 &= \mathcal{M}(u(T)) + \mathcal{E}(u(T)) + \frac{1}{3}\int_{\mathbb{R}\times\mathbb{T}} u^3(T,x,y)\mathrm{d}x\mathrm{d}y \\ &\leqslant \mathcal{M}(u(T)) + \mathcal{E}(u(T)) + 2\,||u(T)||_{L^2}^{3/2}\,||\partial_x u(T)||_{L^2}\,\left|\left|\partial_x^{-1}\partial_y u(T)\right|\right|_{L^2}^{1/2} \\ &\leqslant \mathcal{M}(u(T)) + \mathcal{E}(u(T)) + 2\mathcal{M}(u(T))\,||u(T)||_{\mathbf{E}_\lambda}^2 \end{aligned}$$

Thus, from the conservation of $\mathcal{M}$ and $\mathcal{E}$ (as $u$ is a smooth solution), we finally obtain

$$||u(T)||_{\mathbf{E}_\lambda}^2 \lesssim \mathcal{M}(u_0) + \mathcal{E}(u_0) < +\infty$$

for any $T < T^*$ provided $\epsilon_1^2 < 1/4$, from which we get $T^* = +\infty$.

Finally, let us notice that equation (3.1.6) admits the scaling

$$u_\lambda(t,x,y) := \lambda^{-1}u(\lambda^{-3/2}t, \lambda^{-1/2}x, \lambda^{-1}y),\ (x,y) \in \mathbb{R}\times\mathbb{T}_{\lambda\lambda_0} \tag{3.9.18}$$

meaning that $u_\lambda$ is a solution of (3.1.6) on $[-\lambda^{3/2}T; \lambda^{3/2}T]\times\mathbb{R}\times\mathbb{T}_{\lambda\lambda_0}$ if and only if $u$ is a solution of (3.1.6) on $[-T;T]\times\mathbb{R}\times\mathbb{T}_{\lambda_0}$. Moreover,

$$||u_\lambda(0)||_{\mathbf{E}_{\lambda\lambda_0}} \lesssim \lambda^{-1/4}\,||u(0)||_{\mathbf{E}_{\lambda_0}}$$

Thus, take $u_0 \in \mathbf{E}_{\lambda_0}^\infty$. If $||u_0||_{\mathbf{E}_{\lambda_0}} > \epsilon_1$, then there exists

$$\lambda = \lambda\left(||u_0||_{\mathbf{E}_{\lambda_0}}\right) \sim \epsilon_1^{-4}\,||u_0||_{\mathbf{E}_{\lambda_0}}^{-4} > 1$$

such that $||u_{0,\lambda}||_{\mathbf{E}_{\lambda\lambda_0}^\alpha} \leqslant \epsilon_1$ (since $\epsilon_1 > 0$ is independent of $\lambda \geqslant 1$). Thus, if $u_\lambda \in \mathcal{C}(\mathbb{R}, \mathbf{E}_{\lambda\lambda_0}^\infty)$ is the unique global solution associated with $u_{0,\lambda}$ satisfying (3.9.16) , then

$$u(t,x,y) := \lambda u_\lambda(\lambda^{3/2}t, \lambda^{1/2}x, \lambda y) \in \mathcal{C}\left(\mathbb{R}, \mathbf{E}_{\lambda_0}^\infty\right)$$

is the unique global solution associated with $u_0$.

The rest of the section is devoted to the proof of theorem 3.1.1 (b).

### 3.9.3 Lipschitz bound for the difference of small data solutions

Let $T > 0$, $u_0, v_0 \in \mathbf{E}_\lambda$ and $u, v$ in the class (3.1.8) be the corresponding solutions of the Cauchy problems (3.1.6). As before, up to rescaling and using the conservation of $\mathcal{M}$ and $\mathcal{E}$, it suffices to prove uniqueness for $T = 1$ and

$$||u_0||_{\mathbf{E}_\lambda}, ||v_0||_{\mathbf{E}_\lambda} \leqslant \epsilon_2 \leqslant \epsilon_0$$

Set $w := u - v$. Then $w$ is also in the class (3.9.2) and solves the equation

$$\partial_t w + \partial_x^3 w - \partial_x^{-1}\partial_y w + \partial_x\left(w\cdot\frac{u+v}{2}\right) = 0 \tag{3.9.19}$$

on $[-1;1] \times \mathbb{R} \times \mathbb{T}_\lambda$. Then, since $u_0, v_0$ satisfy (3.9.2), using (3.9.3) and then (3.4.17), (3.5.10)-(3.7.17)-(3.8.23), we obtain for $\epsilon_2$ small enough

$$||w||_{L^\infty_{[-1;1]} L^2_{xy}} \lesssim ||w||_{\overline{\mathbf{F}_\lambda}(1)} \lesssim ||u_0 - v_0||_{L^2} \tag{3.9.20}$$

from which we get $u \equiv v$ on $[-1;1]$ if $u_0 = v_0$.

### 3.9.4 Global well-posedness in the energy space

In this subsection we end the proof of theorem 3.1.1 (b). We proceed as in [IKT08, Section 4].

Take $T > 0$, and let $u_0 \in \mathbf{E}_\lambda$ and $(u_{0,n}) \in (\mathbf{E}_\lambda^\infty)^{\mathbb{N}}$ such that $(u_{0,n})$ converges to $u_0$ in $\mathbf{E}_\lambda$. Again, up to rescaling we can assume $||u_0||_{\mathbf{E}_\lambda} \leqslant \epsilon \leqslant \epsilon_2$ and $||u_{0,n}||_{\mathbf{E}_\lambda} \leqslant \epsilon \leqslant \epsilon_2$. Using again the conservation of $\mathcal{M}$ and $\mathcal{E}$, it then suffices to prove that $(\Phi^\infty(u_{0,n})) \in (\mathcal{C}([-1;1], \mathbf{E}_\lambda^\infty))^{\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([-1;1], \mathbf{E}_\lambda)$.

For a fixed $M > 1$ and $m, n \in \mathbb{N}$, we can split

$$\begin{aligned} ||\Phi^\infty(u_{0,m}) - \Phi^\infty(u_{0,n})||_{L^\infty_1 \mathbf{E}_\lambda} &\leqslant ||\Phi^\infty(u_{0,m}) - \Phi^\infty(P_{\leqslant M} u_{0,m})||_{L^\infty_1 \mathbf{E}_\lambda} \\ &+ ||\Phi^\infty(P_{\leqslant M} u_{0,m}) - \Phi^\infty(P_{\leqslant M} u_{0,n})||_{L^\infty_1 \mathbf{E}_\lambda} + ||\Phi^\infty(P_{\leqslant M} u_{0,n}) - \Phi^\infty(u_{0,n})||_{L^\infty_1 \mathbf{E}_\lambda} \end{aligned}$$

Since

$$||S_T^\infty(P_{\leqslant M} u_{0,n})||_{L^\infty_1 \mathbf{E}^\alpha_\lambda} \leqslant C(\alpha, M)$$

thanks to (3.1.7), the middle term is then controlled with the analogue of (3.9.1) for the difference equation along with a Sobolev inequality with $\alpha$ large enough, which gives

$$||\Phi^\infty(P_{\leqslant M} u_{0,m}) - \Phi^\infty(P_{\leqslant M} u_{0,n})||_{L^\infty_1 \mathbf{E}_\lambda} \leqslant C(M)\, ||u_{0,m} - u_{0,n}||_{\mathbf{E}_\lambda}$$

Therefore it remains to treat the first and last terms. A use of (3.4.16) provides

$$||\Phi^\infty(u_{0,m}) - \Phi^\infty(P_{\leqslant M} u_{0,m})||_{L^\infty_1 \mathbf{E}_\lambda} \lesssim ||\Phi^\infty(u_{0,m}) - \Phi^\infty(P_{\leqslant M} u_{0,m})||_{\mathbf{F}_\lambda(1)}$$

and thus we have to estimate difference of solutions in $\mathbf{F}_\lambda(1)$. Let us write $u_1 := \Phi^\infty(u_{0,m})$, $u_2 := \Phi^\infty(P_{\leqslant M} u_{0,m})$ and $v := u_1 - u_2$.

Using (3.5.2) and (3.7.15) combined with (3.9.3) we obtain the bound

$$||v||_{\mathbf{F}_\lambda(1)} \lesssim ||v||_{\mathbf{B}_\lambda(1)} + ||v||_{\mathbf{F}_\lambda(1)}\, \epsilon$$

Therefore, taking $\epsilon$ small enough, it suffices to control $||v||_{\mathbf{B}_\lambda(1)}$. Using the definition of $\mathbf{B}_\lambda(1)$ (3.4.4), we see that

$$||v||_{\mathbf{B}_\lambda(1)} \leqslant ||P_{\leqslant 1} v_0||_{\mathbf{E}_\lambda} + ||P_{\geqslant 2} v||_{\mathbf{B}_\lambda(1)}$$

Now, in view of the definition of $\mathbf{B}_\lambda(1)$ and $\overline{\mathbf{B}_\lambda}(1)$, we have

$$||P_{\geqslant 2} v||_{\mathbf{B}_\lambda(1)} \sim ||\partial_x P_{\geqslant 2} v||_{\overline{\mathbf{B}_\lambda}(1)} + \left|\left|\partial_x^{-1} \partial_y P_{\geqslant 2} v\right|\right|_{\overline{\mathbf{B}_\lambda}(1)}$$

Combining this remark with the previous estimates, we finally get the bound

$$||v||_{\mathbf{F}_\lambda(1)} \lesssim ||v_0||_{\mathbf{E}_\lambda} + ||P_{\geqslant 2} \partial_x v||_{\overline{\mathbf{B}_\lambda}(1)} + \left|\left|P_{\geqslant 2} \partial_x^{-1} \partial_y v\right|\right|_{\overline{\mathbf{B}_\lambda}(1)} \tag{3.9.21}$$

We now define $U := P_{High}\partial_x v$ and $V := P_{High}\partial_x^{-1}\partial_y v$. We begin by writing down the equations satisfied by $U$ and $V$ :

$$\partial_t U + \partial_x^3 U - \partial_x^{-1}\partial_y^2 U = P_{High}(-u_1 \cdot \partial_x U) + P_{High}(-P_{Low}u_1 \cdot \partial_x^2 P_{Low}v) \\ + P_{High}(-P_{High}u_1 \cdot \partial_x^2 P_{Low}v) + P_{High}(-\partial_x v \cdot \partial_x(u_1+u_2)) + P_{High}(-v \cdot \partial_x^2 u_2) \quad (3.9.22)$$

and

$$\partial_t V + \partial_x^3 V - \partial_x^{-1}\partial_y^2 V = P_{High}(-u_1 \cdot \partial_x V) + P_{High}(-P_{Low}u_1 \cdot \partial_x P_{Low}\partial_x^{-1}\partial_y v) \\ + P_{High}(-P_{High}u_1 \cdot \partial_x P_{Low}\partial_x^{-1}\partial_y v) + P_{High}(-v \cdot \partial_y u_2) \quad (3.9.23)$$

Let us look at (3.9.22). We set $h := -P_{Low}u_1 \cdot \partial_x^2 P_{Low}v$, $w_1 := -P_{High}u_1$, $w_1' := \partial_x^2 P_{Low}v$, $w_2 := -\partial_x v$, $w_2' := \partial_x(u_1+u_2)$ and $w_3 := -v$, $w_3' := \partial_x^2 u_2$. Since $u_1, u_2 \in \mathbf{F}_\lambda(1)$ we have $v \in \mathbf{F}_\lambda(1)$, thus $h$, $w_i$ and $w_i'$ satisfy the assumptions of (3.8.25). Thence we infer

$$||U||^2_{\overline{\mathbf{B}_\lambda}(1)} \lesssim ||\partial_x v_0||^2_{L^2_\lambda} + ||u_1||_{\mathbf{F}_\lambda(1)} ||U||^2_{\overline{\mathbf{F}_\lambda}(1)} \\ + ||U||_{\overline{\mathbf{F}_\lambda}(1)} \Big( ||P_{High}u_1||_{\overline{\mathbf{F}_\lambda}(1)} \left|\left|\partial_x^2 P_{Low}v\right|\right|_{\overline{\mathbf{F}_\lambda}(1)} \\ + ||\partial_x v||_{\overline{\mathbf{F}_\lambda}(1)} ||\partial_x(u_1+u_2)||_{\overline{\mathbf{F}_\lambda}(1)} + ||v||_{\overline{\mathbf{F}_\lambda}(1)} \left|\left|\partial_x^2 u_2\right|\right|_{\overline{\mathbf{F}_\lambda}(1)} \Big)$$

Therefore, using (3.9.3) and (3.9.20), the previous estimate reads

$$||U||^2_{\overline{\mathbf{B}_\lambda}(1)} \lesssim ||v_0||^2_{\mathbf{E}_\lambda} + \epsilon ||U||^2_{\overline{\mathbf{F}_\lambda}(1)} + ||U||_{\overline{\mathbf{F}_\lambda}(1)} \Big( \epsilon ||v_0||_{L^2_\lambda} \\ + ||v||_{\mathbf{F}_\lambda(1)} \epsilon + ||v_0||_{L^2_\lambda} ||u_2||_{\mathbf{F}^2_\lambda(1)} \Big)$$

Proceeding similarly for $V$, we obtain the estimate

$$||V||^2_{\overline{\mathbf{B}_\lambda}(1)} \lesssim \left|\left|\partial_x^{-1}\partial_y v_0\right|\right|^2_{L^2_\lambda} + ||u_1||_{\mathbf{F}_\lambda(1)} ||V||^2_{\overline{\mathbf{F}_\lambda}(1)} + ||V||_{\overline{\mathbf{F}_\lambda}(1)} \\ \cdot \Big( ||P_{High}u_1||_{\overline{\mathbf{F}_\lambda}(1)} \left|\left|\partial_x P_{Low}\partial_x^{-1}\partial_y v\right|\right|_{\overline{\mathbf{F}_\lambda}(1)} + ||v||_{\overline{\mathbf{F}_\lambda}(1)} \left|\left|\partial_x \partial_x^{-1}\partial_y u_2\right|\right|_{\overline{\mathbf{F}_\lambda}(1)} \Big)$$

after applying (3.8.25). Again, a use of (3.9.3) and (3.9.20) gives

$$||V||^2_{\overline{\mathbf{B}_\lambda}(1)} \lesssim ||v_0||^2_{\mathbf{E}_\lambda} + \epsilon ||V||^2_{\overline{\mathbf{F}_\lambda}(1)} + ||V||_{\overline{\mathbf{F}_\lambda}(1)} \Big( \epsilon ||v||_{\mathbf{F}_\lambda(1)} + ||v_0||_{L^2_\lambda} ||u_2||_{\mathbf{F}^2_\lambda(1)} \Big)$$

Combining the estimates for $U$ and $V$ along with (3.9.21), we get the final bound

$$||v||_{\mathbf{F}_\lambda(1)} \lesssim ||v_0||_{\mathbf{E}_\lambda} + \epsilon ||v||_{\mathbf{F}_\lambda(1)} + ||P_{\leqslant M}u_{0,m}||_{\mathbf{E}^2_\lambda} ||v_0||_{L^2_\lambda}$$

since $||u_2||_{\mathbf{F}^2_\lambda(1)} \lesssim ||u_2(0)||_{\mathbf{E}^2_\lambda}$ by (3.9.3).

Taking $\epsilon$ small enough and $M > 1$ large enough concludes the proof.

## 3.10 Failure of uniform continuity for the flow

In this section, we continue the analysis initiated at section 3.3 by constructing a sequence of solutions "responsible" of the quasilinear behaviour, thus showing that the regularity of the flow map obtained in section 3.9.4 above is sharp. Here is the precise statement of proposition 2.1.4 :

**Proposition 3.10.1**

*There exists two positive constants $c, C > 0$ and two sequences $(u_n)$ and $(\widetilde{u_n})$ of solutions to (3.1.6) in the space $\mathcal{C}([-1;1], \mathbf{E}(\mathbb{R}\times\mathbb{T}))$, satisfying for any $t \in [-1;1]$,*

$$\sup_n ||u_n(t)||_{\mathbf{E}} + \sup_n ||\widetilde{u_n}(t)||_{\mathbf{E}} \leqslant C \tag{3.10.1}$$

*and at $t = 0$*

$$\lim_{n\to+\infty} ||u_n(0) - \widetilde{u_n}(0)||_{\mathbf{E}} = 0 \tag{3.10.2}$$

*but such that for any $t \in [-1;1]$,*

$$\liminf_{n\to+\infty} ||u_n(t) - \widetilde{u_n}(t)||_{\mathbf{E}} \geqslant c|t| \tag{3.10.3}$$

**Proof :**
At section 3.3, we saw that the smoothness of the flow map was limited by the resonant low-high interaction. Using a finer analysis of this interaction, Koch et Tzvetkov [KT08] displayed a family of solutions on which the flow map does not act uniformly continuously. The goal of this section is then to construct similar solutions which are periodic in the $y$ variable.

Following the idea in [KT08], we perturb the plane wave $\cos(\omega(\xi_0, q_0)t + \xi_0 x + q_0 y)$ (solution of the linear problem, with $\omega$ defined in (3.10.5)) to get an approximate solution of (3.1.6). The choice of $\xi_0$ and $q_0$ is made so that $\partial_\xi\omega(\xi_0, q_0) = 0$.

First, let $N \in \mathbb{N}$ be a large integer and $\lambda := \sqrt[4]{3}N$. To construct a solution of finite energy, we will apply a spatial cutoff to the plane wave above. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, [0;1])$ be a smooth cutoff function satisfying supp$\chi \subset [-2;2]$ and $\chi \equiv 1$ on $[-1;1]$. For our approximate solution to have an atiderivative, we define then the cutoff $\psi(x) = \chi(x) - \chi(x + x_\lambda)$ where $x_\lambda = 10\sqrt{3}\lambda\pi$ in order to decouple the two pieces. Let us finally define our family of approximate solutions

$$u_\theta(t,x,y) := \lambda^{-3/2}\psi_\lambda(x)\cos\left(\Phi_{\lambda,\theta}(t,x,y)\right) - \lambda^{-1}\theta\widetilde{\psi_\lambda}(x) \tag{3.10.4}$$

where $\widetilde{\psi_\lambda}$ is a cutoff with the same properties as $\psi_\lambda$ with the extra assumption $\widetilde{\psi_\lambda} \equiv 1$ on supp$\psi_\lambda$, and where the phase function is a perturbation of the phase for a plane wave :

$$\Phi_{\lambda,\theta}(t,x,y) = 4\lambda^3 t + \lambda x + \sqrt{3}\lambda^2 y + \theta t, \quad (t,x,y) \in [-1;1]\times\mathbb{R}\times\mathbb{T} \tag{3.10.5}$$

with $\theta \in [-1;1]$.

Let us observe that from our choice of $\lambda$ we have $\sqrt{3}\lambda^2 \in \mathbb{N}$. Moreover, by construction of $\psi$,

$$\int_{\mathbb{R}} \psi_\lambda(x)\mathrm{d}x = 0$$

thus $u_\theta(t) \in \mathbf{E}(\mathbb{R}\times\mathbb{T})$ for any $t, \theta \in [-1;1]$. Indeed, straightforward computations (we refer to [KT08] for the details) give $u_\theta(t) = o_{L^2}(1)$, $\partial_x u_\theta(t) = O_{L^2}(1)$ and after integrations by parts

$$\partial_x^{-1}\partial_y u_\theta(t) = \sqrt{3}\lambda^{-1/2}\psi_\lambda\cos\Phi - \sqrt{3}\lambda^{-3/2}\int_{-\infty}^x \psi'_\lambda \sin\Phi\mathrm{d}x' = O_{L^2}(1)$$

uniformly in $t, \theta \in [-1;1]$ and $\lambda$.

Next, using Leibniz rule and integrating by parts, we get

$$\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right)u_\theta = -\theta\lambda^{-3/2}\psi_\lambda\sin\Phi + O(\lambda^{-2})$$

which gives, after using that $\widetilde{\psi} \equiv 1$ on supp$\psi$, that the nonlinear term is finally

$$u_\theta \partial_x u_\theta = \theta \lambda^{-3/2} \psi_\lambda \sin \Phi + O(\lambda^{-3/2})$$

Thus

$$\left|\left|(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2) u_\theta(t) + u_\theta(t)\partial_x u_\theta(t)\right|\right|_{L^2} \lesssim \lambda^{-3/2} \tag{3.10.6}$$

uniformly in $t, \theta, \lambda$. At last, for $\theta_1 = 1$ and $\theta_2 = -1$, we have the lower bound

$$\begin{aligned} ||\partial_x(u_1(t) - u_{-1}(t))||_{L^2} &= \left|\left|\lambda^{-1/2}\psi_\lambda\left(\sin\Phi_1 - \sin\Phi_{-1}\right)\right|\right|_{L^2} + O(\lambda^{-3/2}) \\ &= \left|\left|2\lambda^{-1/2}\psi_\lambda \sin(t)\cos\left(\frac{\Phi_1 + \Phi_{-1}}{2}\right)\right|\right|_{L^2} + O(\lambda^{-3/2}) \\ &\geqslant c|t| + O(\lambda^{-3/2}) \end{aligned} \tag{3.10.7}$$

Finally, we can take for $u_n$ and $\widetilde{u_n}$ the genuine solutions arising from $u_1(0)$ and $u_{-1}(0)$ with $\lambda = \sqrt[4]{3}n$. (3.1.7) shows that they are uniformly bounded in $\mathcal{C}([-1;1], \mathbf{E}_\lambda)$, and they stay close enough to the approximate solutions thanks to (3.10.6) and a standard energy estimate. So we can conclude the proof of proposition 3.10.1 as in [KT08].

□

## 3.11 Orbital stability of the line soliton

In this last section, we turn to the proof of corollary 3.1.2. We briefly recall the main steps of [RT12, Section 2].

Let us remember that equation (3.1.6) has a Hamiltonian structure, with Hamiltonian $E(u)$. To study the orbital stability of $Q_c(x - ct)$, we first make a change of variable to see $Q_c(x)$ as a stationary solution of (3.1.6) rewriten in a moving frame :

$$\partial_t u - c\partial_x u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0 \tag{3.11.1}$$

Equation (3.11.1) still has a Hamitonian structure, with the new Hamiltonian

$$\mathcal{E}_c(u) := \mathcal{E}(u) + c\mathcal{M}(u)$$

The key idea of the proof is then to show, as for the orbital stability of $Q_c$ under the flow of KdV [Ben72], that the Hessian of $\mathcal{E}_c$ about $Q_c$ is strictly positive on the codimension-2 subspace $H := \{\langle v, Q_c\rangle_{L^2} = \langle v, Q_c'\rangle_{L^2} = 0\}$ to get a lower bound on $\mathcal{E}_c(\Phi^1(u_0)(t)) - \mathcal{E}_c(Q_c)$ in term of $\left|\left|\Phi^1(u_0)(t) - Q_c\right|\right|_{\mathbf{E}}$.

To study $D^2\mathcal{E}_c(Q_c)$ on $H$, we begin by computing

$$\begin{aligned} \mathcal{E}_c(Q_c + v(t)) = \mathcal{E}_c(Q_c) + \left( ||\partial_x v||_{L^2}^2 + \left|\left|\partial_x^{-1}\partial_y v\right|\right|_{L^2}^2 + c\,||v||_{L^2}^2 - \int_{\mathbb{R}\times\mathbb{T}} Q_c \cdot v^2 \mathrm{d}x\mathrm{d}y \right) \\ - \int_{\mathbb{R}\times\mathbb{T}} v^3 \mathrm{d}x\mathrm{d}y \end{aligned}$$

The linear term in $v$ vanishes since $Q_c$ is a stationary solution.

Using the Plancherel identity in the $y$ variable, we can write the Hessian of $\mathcal{E}_c$ about $Q_c$ as the sum of the bilinear forms

$$\frac{1}{2}D^2\mathcal{E}_c(Q_c)(v,v) = \sum_{k\in\mathbb{Z}} B_c^k(\mathcal{F}_y v(t,x,k), \mathcal{F}_y v(t,x,k))$$

with

$$B_c^k\left(\widetilde{v}(x), \widetilde{v}(x)\right) = ||\partial_x \widetilde{v}||_{L^2}^2 + k^2 \left|\left|\partial_x^{-1}\widetilde{v}\right|\right|_{L^2}^2 + c\,||\widetilde{v}||_{L^2}^2 - \int_{\mathbb{R}} Q_c \cdot \widetilde{v}^2 \mathrm{d}x$$

Observe that $B_c^0$ is the Hessian about $Q_c$ of the Hamiltonian associated with the KdV equation in a moving frame, and thus by the study in [Ben72] $B_c^0$ is $H^1$ bounded from below as desired.

To treat the terms with $k \neq 0$, first make the change of test function

$$f(x) := \partial_x^{-1}\mathcal{F}_y v(t,x,k) \in L^2(\mathbb{R})$$

Then, using that $k^2 \geqslant 1$, we can write

$$B_c^k(\mathcal{F}_y v(t,x,k), \mathcal{F}_y v(t,x,k)) \geqslant \langle \mathcal{L}_c f, f\rangle$$

where the linear operator $\mathcal{L}_c$ is defined as

$$\mathcal{L}_c := \partial_x^4 - c\partial_x^2 + \partial_x Q_c \partial_x + 1$$

Since $Q_c$ is exponentially decreasing, $\partial_x Q_c \partial_x$ is compact with respect to $\partial_x^4 - c\partial_x^2 + 1$ and thus $\mathrm{Spec}_{ess}\mathcal{L}_c \subset [1, +\infty[$. To get a lower bound on $\langle \mathcal{L}_c f, f\rangle$, it remains to study the existence of negative eigenvalues. Following the method of [APS97], a change of variables leads to consider the eigenvalue problem

$$g^{(4)} - 4\left(1 - \frac{3}{\cosh^2}\right)g'' + 3\nu^2 g = 0 \tag{3.11.2}$$

where

$$3\nu^2 = \frac{16}{c^2}(1-\lambda_0)$$

and $\lambda_0 \leqslant 0$ is the possible negative eigenvalue. Using again the exponential decreasing of $Q_c$, $g$ behaves at infinity as a solution of the linear equation

$$h^{(4)} - 4h'' + 3\nu^2 h = 0 \tag{3.11.3}$$

For each characteristic value $\mu$ of (3.11.3), there is an exact solution

$$g_\mu(x) := \mathrm{e}^{\mu x}\left(\mu^3 + 2\mu - 3\mu^2\tanh(x)\right)$$

of (3.11.2). For these solutions to behave as $\mathrm{e}^{\mu x}$ at infinity, this requires

$$\mu^3 + 2\mu - 3\mu^2 = 0$$

As $\mu$ is also a characteristic value, this implies $\mu = 1$ and thus $\nu^2 = 1$ from which we finally infer

$$c^2 = \frac{16}{3}(1-\lambda_0)$$

Consequently, there is no possible negative eigenvalue $\lambda_0$ if $c < c^* = 4/\sqrt{3}$.

Hence we have a lower $L^2$ bound for the bilinear form associated with $\mathcal{L}_c$, which provides the bound

$$B_c^k(\widetilde{v},\widetilde{v}) \gtrsim \left|\left|\partial_x^{-1}\widetilde{v}\right|\right|_{L^2}^2$$

Linearly interpolating with the obvious bound (since $Q_c \leqslant 3c$)

$$B_c^k(\widetilde{v},\widetilde{v}) \geqslant ||\partial_x\widetilde{v}||_{L^2}^2 + \left|\left|\partial_x^{-1}\widetilde{v}\right|\right|_{L^2}^2 - 2c\,||\widetilde{v}||_{L^2}^2$$

yields to an $L^2$ lower bound for $B_c^k$, which in return provides the final bound

$$B_c^k(\widetilde{v},\widetilde{v}) \gtrsim ||\widetilde{v}||_{H^1}^2 + k^2\left|\left|\partial_x^{-1}\widetilde{v}\right|\right|_{L^2}^2$$

uniformly in $k$.

The last trilinear term $\displaystyle\int v^3$ is treated with the anisotropic Sobolev inequality (3.9.17).

Combining all the bounds from below finally provides a control of $||w||_{\mathbf{E}}$ in term of $\mathcal{E}_c(Q_c + w_0) - \mathcal{E}_c(Q_c)$ for any $w \in H$. The end of the proof is then standard (cf. [Ben72],[RT12, Section 2]).

# Chapter 4

# Study of higher-order KP-I equation on the torus

This chapter essentially contains the paper [Rob17].

## 4.1 Introduction

The KP equations arised in [KP70] as fluid mechanics models for long, weakly nonlinear two-dimensional waves with a small dependence in the tranverse variable. The usual KP equations are

$$\partial_t u + \partial_x^3 u + \epsilon \partial_x^{-1} \partial_y u + u \partial_x u = 0 \tag{4.1.1}$$

where the coefficient $\epsilon$ depends on the surface tension. The KP-I equation corresponds to $\epsilon = -1$, and the KP-II equation to $\epsilon = 1$. The Cauchy problem for these equations has been extensively studied in the past twenty years. The KP-II equation is known to be locally well-posed in the scale-critical space $H^{-1/2,0}(\mathbb{R}^2)$ [HHK09], and globally well-posed in $L^2(\mathbb{R} \times \mathbb{T})$ [MST11] and $L^2(\mathbb{T}^2)$ [Bou93b].

As for the KP-I equation, some ill-posedness results [MST02b, KT08] have shown that this equation does not have a semilinear nature, in the sense that it cannot be treated via a perturbative method. Ionescu, Kenig and Tataru [IKT08] thus developped the short-time Fourier restriction norm method to overcome the resonant low-high interactions responsible of the quasilinear behavior, therefore obtaining global well-posedness in the energy space on $\mathbb{R}^2$. The adaptation [Zha15] in the periodic setting revealed a logarithmic divergence in the energy estimate due to a bad frequency interaction in the resonant set, establishing therefore a local well-posedness result in the Besov space $\mathbf{B}^1_{2,1}(\mathbb{T}^2)$ which is strictly larger than the natural energy space. To overcome this difficulty and recover a global well-posedness result in the energy space, one can look for a better dispersion effect by either removing the assumption of periodicity in one direction [Rob18], or studying higher-order models.

To pursue this latter issue, we investigate the Cauchy problem for the periodic fifth-order KP-I equation

$$\partial_t u - \partial_x^5 u - \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0, \ (t,x,y) \in \mathbb{R} \times \mathbb{T}^2 \tag{4.1.2}$$

First, as noticed by Bourgain [Bou93b] in the context of the periodic KP-II equation, any (periodic in space) solution of (4.1.2) has a constant (in $y$) $x$-mean value, i.e if $(m,n) \in \mathbb{Z}^2$ are the

Fourier variables associated with $(x, y) \in \mathbb{T}^2$, then the Fourier coefficients of $u$ with respect to $(x, y)$ satisfy the extra condition

$$\widehat{u}(t, 0, n) = 0 \text{ for } n \in \mathbb{Z} \setminus \{0\} \tag{4.1.3}$$

In particular, in $t = 0$ we see that the initial data must satisfy (4.1.3). As in [Bou93a, Bou93b], we will make the additional assumption that $\widehat{u_0}(0, 0) = 0$, which is not restrictive since for data $u_0$ with non zero constant $x$ mean value, we will just have to set $v_0 := u_0 - \widehat{u_0}(0, 0)$ which satisfies the above condition and the modified equation

$$\partial_t v - \partial_x^5 v + c\partial_x v - \partial_x^{-1}\partial_y^2 v + v\partial_x v = 0$$

with $c = \widehat{u_0}(0, 0)$. Our analysis of (4.1.2) applies equally to the above modified equation, since the extra lower order term does not change the resonant function (see its definition in (4.3.7) below).

Now, to work with initial data satisfying the constraint (4.1.3), we introduce the subspace of distributions

$$\mathcal{D}'_0(\mathbb{T}^2) := \left\{u_0 \in \mathcal{D}'(\mathbb{T}^2),\ \widehat{u_0}(0, n) = 0\ \forall n \in \mathbb{Z}\right\}$$

in which the operator $\partial_x^{-1}$ is well defined as

$$\partial_x^{-1} u_0(x, y) := \mathcal{F}^{-1}\left\{\frac{1}{im}\widehat{u_0}(m, n)\right\}$$

The equation (4.1.2) has another interesting feature : it possesses some conservation laws. Indeed, the mass

$$\mathcal{M}(u_0) := \int_{\mathbb{T}^2} u_0^2(x, y)\mathrm{d}x\mathrm{d}y \tag{4.1.4}$$

and the energy

$$\mathcal{E}(u_0) := \int_{\mathbb{T}^2}\left\{(\partial_x^2 u_0)^2(x, y) + (\partial_x^{-1}\partial_y u_0)^2(x, y) - \frac{1}{3}u_0^3(x, y)\right\}\mathrm{d}x\mathrm{d}y \tag{4.1.5}$$

are conserved by the flow. Therefore, to obtain a global well-posedness result, it suffices to construct local solutions to (4.1.2) and they will be automatically extended globally in time as soon as the above quantities are bounded.

In view of the precedent remarks, we will thus work in the energy space defined as

$$\mathbf{E}(\mathbb{T}^2) := \left\{u_0 \in \mathcal{D}'_0(\mathbb{T}^2) \cap L^2(\mathbb{T}^2),\ \partial_x^2 u_0 \in L^2(\mathbb{T}^2),\ \partial_x^{-1}\partial_y u_0 \in L^2(\mathbb{T}^2)\right\} \tag{4.1.6}$$

endowed with the norm

$$||u_0||_{\mathbf{E}} := \left(||u_0||_{L^2}^2 + \left|\left|\partial_x^2 u_0\right|\right|_{L^2}^2 + \left|\left|\partial_x^{-1}\partial_y u_0\right|\right|_{L^2}^2\right)^{1/2}$$

For initial data in this space, the mass is clearly finite, and due to the anisotropic Sobolev inequality (3.9.17) of Tom [Tom96, Lemma 2.5] the energy is bounded as well.

The first results on the Cauchy problem for (4.1.2) were obtained by Iório and Nunes in the general setting of [IN98] where it has been shown to be locally well-posed for zero mean value initial data in the space $H^s(\mathbb{T}^2)$ for $s > 2$ by adapting the general quasi-linear theory of Kato. This model has then been studied in the work of Saut and Tzvetkov [ST99, ST00, ST01], where it has been proved that this equation is globally well-posed in the energy spaces $\mathbf{E}(\mathbb{R}^2)$ and $\mathbf{E}(\mathbb{T} \times \mathbb{R})$ by using the standard Bourgain method. Li and Xiao [LX08] have then pushed forward with this approach and got global well-posedness in $L^2(\mathbb{R}^2)$. However, a counter-example is built in [ST01] to show the failure of the bilinear estimate in the usual Bourgain spaces when $u$ is periodic in both variables, initiating thereafter a systematic study of such quasilinear behaviours in dispersive equations (see [Tzv04] for a detailed presentation of this issue). This implies that another approach is needed. Using the refined energy method of [KT03], Ionescu and Kenig [IK07] proved global well-posedness in $\mathbf{E}(\mathbb{R}\times\mathbb{T})$. Very lately, Guo, Huo and Fang [GHF17] proved local well-posedness in $H^{s,0}(\mathbb{R}^2)$ for $s \geqslant -3/4$ and the initial-value problem (4.1.2) for periodic initial data in the energy space remained open. In this note, we prove the following.

**Theorem 4.1.1**

*(a) For any* $u_0 \in \mathbf{E}^\infty(\mathbb{T}^2)$, *there exists a unique global smooth solution*

$$u =: \Phi^\infty(u_0) \in \mathcal{C}(\mathbb{R}, \mathbf{E}^\infty(\mathbb{T}^2))$$

*to (4.1.2) and moreover, for any* $T > 0$ *and* $\sigma \geqslant 2$ *we have*

$$||\Phi^\infty(u_0)||_{L^\infty_T \mathbf{E}^\sigma} \leqslant C\left(T, \sigma, ||u_0||_{\mathbf{E}^\sigma}\right) \tag{4.1.7}$$

*(b) Take any* $u_0 \in \mathbf{E}(\mathbb{T}^2)$ *and* $T > 0$, *then there exists a unique solution* $u$ *to (4.1.2) in the class*

$$\mathcal{C}\left([-T;T], \mathbf{E}\right) \cap \mathbf{F}(T) \cap \mathbf{B}(T) \tag{4.1.8}$$

*This defines a continuous flow* $\Phi : \mathbf{E} \to \mathcal{C}(\mathbb{R}, \mathbf{E})$ *which leaves* $\mathcal{M}$ *and* $\mathcal{E}$ *invariants.*

The functions spaces $\mathbf{E}^\infty$, $\mathbf{F}(T)$ and $\mathbf{B}(T)$ are defined in section 4.2 below.

Now, in view of the above definition of the energy space, one may be surprised by the gap in regularity between the Cauchy theory in $\mathbb{R}^2$ [GHF17] and our well-posedness result. This is explained by the difficulty to evaluate accurately the measure of the resonant set in the periodic setting. See remark 4.3.7 below for more details.

To prove Theorem 4.1.1, we will then use the method of [IKT08] and prove the linear, bilinear and energy estimates in the spaces $\mathbf{F}, \mathbf{N}$ and $\mathbf{B}$.

Section 4.2 introduces general functions spaces and their basic properties. We prove some dyadic estimates in section 4.3 which we will use in sections 4.4 and 4.5 to prove energy and bilinear estimates respectively. The proof of Theorem 4.1.1 is finally completed in section 4.6. As a final comment, we give the proof of proposition 2.2.1 at then end of section 4.6.

**Notations**

For positive real numbers $a$ and $b$, $a \lesssim b$ means that there exists a positive constant $c > 0$ (independant of the various parameters) such that $a \leqslant c \cdot b$.
The notation $a \sim b$ stands for $a \lesssim b$ and $b \lesssim a$.
For $x \in \mathbb{R}^d$ we set $\langle x \rangle := (1 + |x|^2)^{1/2}$.
For a set $A \subset \mathbb{R}^d$, $\mathbb{1}_A$ is the characteristic function of $A$ and if $A$ is Lebesgue-measurable, $|A|$ means its measure. When $A \subset \mathbb{Z}$ is a finite set, its cardinal is denoted $\#A$.
For $M > 0$ and $s \in \mathbb{R}$, $\lesssim M^{s-}$ means $\leqslant C_\varepsilon M^{s-\varepsilon}$ for any choice of $\varepsilon > 0$ small enough. We define similarly $M^{s+}$.
Let $(\tau, m, n) \in \mathbb{R}\times\mathbb{Z}^2$ denote the Fourier variables of $(t, x, y) \in \mathbb{R}\times\mathbb{T}^2$. We define the unitary group

$$U(t) = \mathrm{e}^{-t(\partial_x^5 + \partial_x^{-1}\partial_y^2)} = \mathcal{F}_{xy}^{-1}\mathrm{e}^{-it\omega(m,n)}\mathcal{F}_{xy}$$

where $\omega(m,n) := m^5 + \frac{n^2}{m}$.
We note $M, K \in 2^{\mathbb{N}}$ the dyadic frequency decompositions of $|m|$ and $\langle \tau + \omega(m,n)\rangle$. We define then $D_{M,K} := \left\{(\tau,m,n) \in \mathbb{R}\times\mathbb{Z}^2,\quad |m|\sim M, \langle \tau+\omega(m,n)\rangle \sim K\right\}$ and $D_{M,\leqslant K} := \bigcup_{K'\leqslant K} D_{M,K'}$.
We note also $I_M := \{5M/8 \leqslant |m| \leqslant 8M/5\}$ and $I_{\leqslant M} := \bigcup_{M'\leqslant M} I_{M'}$.
We use the notations $M_1 \wedge M_2 := \min(M_1, M_2)$ and $M_1 \vee M_2 := \max(M_1, M_2)$.
For $M_1, M_2, M_3 \in \mathbb{R}_+^*$, $M_{\min} \leqslant M_{med} \leqslant M_{max}$ denotes the increasing rearrangement of $M_1, M_2, M_3$, i.e

$$M_{min} := M_1 \wedge M_2 \wedge M_3, \quad M_{max} = M_1 \vee M_2 \vee M_3 \\ \text{and } M_{med} = M_1 + M_2 + M_3 - M_{max} - M_{min}$$

We define now the Littlewood-Paley decomposition. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with $0 \leqslant \chi \leqslant 1$, $\operatorname{supp}\chi \subset [-8/5; 8/5]$ and $\chi \equiv 1$ on $[-5/4; 5/4]$.
For $K \in 2^{\mathbb{N}}$, we then define $\eta_1(x) := \chi(x)$ and $\eta_K(x) := \chi(x/K) - \chi(2x/K)$ if $K > 1$, such that $\operatorname{supp}\eta_K \subset I_K$ and $\eta_K \equiv 1$ on $\{4/5K \leqslant |x| \leqslant 5/4K\}$ for $K > 1$. Thus $\langle \tau+\omega(m,n)\rangle \in \operatorname{supp}\eta_K \Rightarrow \langle\tau+\omega\rangle \in I_K$ and $|\tau+\omega| \sim K$ for any $K \in 2^{\mathbb{N}}$.
When needed, we may use another decomposition $\widetilde{\chi}$, $\widetilde{\eta}$ with the same properties as $\chi$, $\eta$ and satisfying $\widetilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$ and $\widetilde{\eta} \equiv 1$ on $\operatorname{supp}\eta$.
Finally, for $\kappa \in \mathbb{R}_+^*$, we note $\chi_\kappa(x) := \chi(x/\kappa)$.

We also define the Littlewood-Paley projectors associated with the sets $I_M$ :

$$P_M u := \mathcal{F}^{-1}\left(\mathbb{1}_{I_M}(m)\widehat{u}\right) \text{ and } P_{\leqslant M} u := \sum_{M'\leqslant M} P_M u = \mathcal{F}^{-1}\left(\mathbb{1}_{I_{\leqslant M}}(m)\widehat{u}\right)$$

## 4.2 Functions spaces and first properties

### 4.2.1 Definitions

The energy space $\mathbf{E}$ was defined in (4.2.5). More generally, for $\sigma \geqslant 2$, we define

$$\mathbf{E}^\sigma(\mathbb{T}^2) := \left\{u_0 \in \mathfrak{D}_0'(\mathbb{T}^2) \cap L^2(\mathbb{T}^2),\ ||u_0||_{\mathbf{E}^\sigma} := ||\langle m\rangle^\sigma \cdot p(m,n) \cdot \widehat{u_0}||_{L^2} < +\infty\right\} \tag{4.2.1}$$

and

$$\mathbf{E}^\infty = \bigcap_{\sigma\geqslant 2} \mathbf{E}^\sigma$$

with the weight $p$ defined as

$$p(m,n) := \left\langle \langle m\rangle^{-2}\frac{n}{m}\right\rangle,\ (m,n) \in (\mathbb{Z}^*)^2$$

so that with this definition $\mathbf{E} = \mathbf{E}^2$.

Let $M \in 2^{\mathbb{N}}$. As in [IKT08], for $b \in [0; 1/2]$ the dyadic Bourgain type space is defined as

$$X_M^b := \Big\{f(\tau,m,n) \in L^2(\mathbb{R}\times\mathbb{Z}^2), \operatorname{supp}f \subset \mathbb{R}\times I_M \times \mathbb{Z}, \\ ||f||_{X_M^b} := \sum_{K\geqslant 1} K^b\, ||\rho_K(\tau+\omega)f||_{L^2} < +\infty\Big\}$$

When $b = 1/2$ we simply write $X_M$.

Then, we use the $X_M^b$ structure uniformly on time intervals of size $M^{-2}$ :

$$F_M^b := \Big\{ u(t,x,y) \in \mathcal{C}\left(\mathbb{R}, \mathbf{E}^\infty\right),\ P_M u = u, \\ ||u||_{F_M^b} := \sup_{t_M \in \mathbb{R}} ||p \cdot \mathcal{F}\left\{\chi_{M^{-2}}(t - t_M) u\right\}||_{X_M^b} < +\infty \Big\}$$

and

$$N_M := \Big\{ u(t,x,y) \in L^2\left(\mathbb{R}, \mathbf{E}^\infty\right),\ P_M u = u, \\ ||u||_{N_M} := \sup_{t_M \in \mathbb{R}} \left|\left| p \cdot \left|\tau + \omega + iM^2\right|^{-1} \mathcal{F}\left\{\chi_{M^{-2}}(t - t_M) u\right\} \right|\right|_{X_M} < +\infty \Big\}$$

For a function space $Y \hookrightarrow \mathcal{C}(\mathbb{R}, \mathbf{E}^\infty)$, we set

$$Y(T) := \Big\{ u \in \mathcal{C}\left([-T,T], \mathbf{E}^\infty\right),\ ||u||_{Y(T)} < +\infty \Big\}$$

endowed with

$$||u||_{Y(T)} := \inf\left\{||\widetilde{u}||_Y\,,\quad \widetilde{u} \in Y,\quad \widetilde{u} \equiv u \text{ on } [-T,T]\right\} \tag{4.2.2}$$

Finally, the main function spaces are defined as

$$\mathbf{F}^{\sigma,b}(T) := \left\{ u \in \mathcal{C}([-T,T], \mathbf{E}^\sigma),\ ||u||_{\mathbf{F}^{\sigma,b}(T)} := \left( \sum_{M \geqslant 1} M^4 \, ||P_M u||^2_{F_M^b(T)} \right)^{1/2} < +\infty \right\} \tag{4.2.3}$$

and

$$\mathbf{N}^{\sigma}(T) := \left\{ u \in L^2([-T,T], \mathbf{E}^\sigma),\ ||u||_{\mathbf{N}^{\sigma}(T)} := \left( \sum_{M \geqslant 1} M^4 \, ||P_M u||^2_{N_M(T)} \right)^{1/2} < +\infty \right\} \tag{4.2.4}$$

The last space is the energy-type space which is the analogue in this context of the usual space $L^\infty([-T;T], \mathbf{E}^\sigma)$ :

$$\mathbf{B}^\sigma(T) := \Big\{ u \in \mathcal{C}([-T,T], \mathbf{E}^\sigma), \\ ||u||_{\mathbf{B}^\sigma(T)} := \left( \sum_{M \geqslant 1} \sup_{t_M \in [-T,T]} ||P_M u(t_M)||^2_{\mathbf{E}^\sigma} \right)^{1/2} < +\infty \Big\} \tag{4.2.5}$$

Again, for $F_M^b$ and $\mathbf{F}^{\sigma,b}(T)$, if $b = 1/2$ we just drop it. We do the same for $\sigma = 2$.

For the difference equation, we use similar spaces $\overline{F_M}$, $\overline{N_M}$ and $\overline{\mathbf{F}}(T)$, $\overline{\mathbf{N}}(T)$ and $\overline{\mathbf{B}}(T)$ which are the same as the above spaces but without the weight $p$ and at regularity $\sigma = 0$. Let us notice that in view of the definition of $p$ we then have

$$||u||^2_{F_M(T)} \sim ||u||^2_{\overline{F_M}(T)} + M^{-4} \left|\left|\partial_x^{-1} \partial_y u\right|\right|^2_{\overline{F_M}(T)}$$

### 4.2.2 Basic properties

We collect here some basic properties of the spaces $X_M$, $\mathbf{F}(T)$ and $\mathbf{N}(T)$. The proof of these results can be found e.g in [IKT08, GO16, KP15, Rob18].

First, for any $f_M \in X_M$, we have

$$||f_M||_{\ell^2_{m,n}L^1_\tau} \lesssim ||f_M||_{X_M} \tag{4.2.6}$$

Moreover, if we take $\gamma \in L^2(\mathbb{R})$ satisfying

$$|\widehat{\gamma(\tau)}| \lesssim \langle\tau\rangle^{-4} \tag{4.2.7}$$

then for any $K_0 \geqslant 1$ and $t_0 \in \mathbb{R}$ we have

$$K_0^{1/2} \left|\left|\chi_{K_0}(\tau+\omega)\mathcal{F}\left\{\gamma(K_0(t-t_0))\mathcal{F}^{-1}f_M\right\}\right|\right|_{L^2} + \sum_{K\geqslant K_0} K^{1/2} \left|\left|\rho_K(\tau+\omega)\mathcal{F}\left\{\gamma(K_0(t-t_0))\mathcal{F}^{-1}f_M\right\}\right|\right|_{L^2} \lesssim ||f_M||_{X_M} \tag{4.2.8}$$

and the implicit constants are independent of $M$, $K_0$ and $t_0$.

For general time multipliers $m_M \in \mathcal{C}^4(\mathbb{R})$ bounded along with its derivatives, as in [IKT08] we have the bounds

$$||m_M(t)f_M||_{F_M} \lesssim \left(\sum_{k=0}^{4}(1\vee M)^{-k}\left|\left|m_M^{(k)}\right|\right|_{L^\infty}\right)||f_M||_{F_M} \tag{4.2.9}$$

and

$$||m_M(t)f_M||_{N_M} \lesssim \left(\sum_{k=0}^{4}(1\vee M)^{-k}\left|\left|m_M^{(k)}\right|\right|_{L^\infty}\right)||f_M||_{N_M^{b,b_1}} \tag{4.2.10}$$

We will also use [GO16, Lemma 3.4] to get a factor $T^{0+}$ in the estimates in order to avoid rescaling :

**Lemma 4.2.1**

*Let $T \in ]0;1]$ and $0 \leqslant b < 1/2$. Then, for any $u \in F_M(T)$,*

$$||u||_{F_M^b(T)} \lesssim T^{(1/2-b)-}\,||u||_{F_M(T)} \tag{4.2.11}$$

*and the implicit constant is independent of $M$ and $T$.*

The last estimate justifies the use of $\mathbf{F}(T)$ as a resolution space :

**Lemma 4.2.2**

*Let $\sigma \geqslant 2$, $T \in ]0;1]$ and $u \in \mathbf{F}^\sigma(T)$. Then*

$$||u||_{L^\infty_T\mathbf{E}^\sigma} \lesssim ||u||_{\mathbf{F}^\sigma(T)} \tag{4.2.12}$$

### 4.2.3 Linear estimate

In this last subsection, we recall a linear estimate which replaces the usual estimate in the context of standard Bourgain spaces. The proof is the same as the one of [IKT08, Proposition 3.2].

**Proposition 4.2.3**

*Let $T>0$ and $u, f \in \mathcal{C}([-T;T], \mathbf{E}^\infty)$ satisfying*

$$\partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u = f \tag{4.2.13}$$

*on $[-T,T]\times\mathbb{T}^2$.*
*Then for any $\sigma \geqslant 2$, we have*

$$||u||_{\mathbf{F}^\sigma(T)} \lesssim ||u||_{\mathbf{B}^\sigma(T)} + ||f||_{\mathbf{N}^\sigma(T)} \tag{4.2.14}$$

*and*

$$||u||_{\overline{\mathbf{F}}(T)} \lesssim ||u||_{\overline{\mathbf{B}}(T)} + ||f||_{\overline{\mathbf{N}}(T)} \tag{4.2.15}$$

## 4.3 Dyadic estimates

We prove here several estimates on the trilinear form $\int_{\mathbb{R}}\sum_{\mathbb{Z}^2} f_1 \star f_2 \cdot f_3$ which replace [IKT08, Corollary 5.3] in our context.

For the proof of the following easy lemmas, we refer to [Rob18, Section 3].

**Lemma 4.3.1**

*Let $f_i \in L^2(\mathbb{R}\times\mathbb{Z}^2)$ be such that $\operatorname{supp} f_i \subset D_{M_i,\leqslant K_i} \cap \mathbb{R}\times\mathbb{Z}\times I_i$, with $M_i, K_i \in 2^{\mathbb{N}}$ and $I_i \subset \mathbb{Z}$, $i=1,2,3$. Then*

$$\int_{\mathbb{R}}\sum_{\mathbb{Z}^2} f_1 \star f_2 \cdot f_3 \lesssim M_{min}^{1/2} K_{min}^{1/2} (\# I_{min})^{1/2} \prod_{i=1}^{3} ||f_i||_{L^2} \tag{4.3.1}$$

**Lemma 4.3.2**

*Let $\Lambda \subset \mathbb{Z}^2$. We assume that the projection of $\Lambda$ on the $m$ axis is contained in an interval $I \subset \mathbb{Z}$. Moreover, we assume that the cardinal of the $n$-sections of $\Lambda$ (that is the sets $\{n\in\mathbb{Z}, (m_0,n)\in\Lambda\}$ for a fixed $m_0$) is uniformly (in $m_0$) bounded by a constant $C$. Then we have*

$$|\Lambda| \leqslant C\langle |I|\rangle$$

**Lemma 4.3.3**

*Let $I$, $J$ be two intervals in $\mathbb{R}$, and let $\varphi : I \to \mathbb{R}$ be a $\mathcal{C}^1$ function with $\inf_{x\in J}|\varphi'(x)| > 0$. Assume that $\{n \in J\cap\mathbb{Z},\ \varphi(n)\in I\} \neq \emptyset$. Then*

$$\#\{n \in J\cap\mathbb{Z},\ \varphi(n)\in I\} \lesssim \left\langle \frac{|I|}{\inf_{x\in J}|\varphi'(x)|}\right\rangle \tag{4.3.2}$$

**Lemma 4.3.4**

*Let $a\neq 0$ ,$b$, $c$ be real numbers and $I\subset\mathbb{R}$ a bounded interval. Then*

$$\#\left\{n\in\mathbb{Z},\ an^2+bn+c\in I\right\} \lesssim \left\langle \frac{|I|^{1/2}}{|a|^{1/2}}\right\rangle \tag{4.3.3}$$

The main estimates of this section are the following :

**Proposition 4.3.5**

*Let $M_i, K_i \in 2^{\mathbb{N}}$, $i = 1,2,3$, and take $u_1, u_2 \in L^2(\mathbb{R}\times\mathbb{Z}^2)$ be such that* $\mathrm{supp}(u_i) \subset D_{M_i,\leqslant K_i}$*. Then*

$$\left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot u_1 \star u_2\right|\right|_{L^2} \lesssim (K_1\wedge K_2)^{1/2} M_{min}^{1/2} \cdot \left\langle (K_1\vee K_2)^{1/4}(M_1\wedge M_2)^{1/4}\right\rangle ||u_1||_{L^2}\,||u_2||_{L^2} \quad (4.3.4)$$

*Moreover, if we are in the case $K_{max} \leqslant 10^{-10} M_1M_2M_3M_{max}^2$, then*

$$\left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot u_1 \star u_2\right|\right|_{L^2} \lesssim (K_1\wedge K_2)^{1/2} M_{min}^{1/2} \cdot \left\langle (K_1\vee K_2)^{1/2}(M_3M_{max})^{-1/2}\right\rangle ||u_1||_{L^2}\,||u_2||_{L^2} \quad (4.3.5)$$

**Proof :**

These estimates are the analogue of those proved in [ST01, Subsections 2.1& 2.2] in the context of the bilinear estimate in standard Bourgain spaces. The proof is very similar to that of [Rob18, Proposition 5.5]. First, we split $u_1$ and $u_2$ depending on the value of $m_i$ on an $M_3$ scale, meaning

$$\left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot u_1 \star u_2\right|\right|_{L^2} \leqslant \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot u_{1,k} \star u_{2,j}\right|\right|_{L^2} \quad (4.3.6)$$

with

$$u_{i,j} := \mathbb{1}_{[jM_3,(j+1)M_3]}(m_i)u_i$$

The conditions $|m| \sim M_3$, $m_1 \in [kM_3,(k+1)M_3]$ and $m-m_1 \in [jM_3;(j+1)M_3]$ require $j \in [-k-c;-k+c]$ for an absolute constant $c>0$.

Squaring the norm in the right-hand side of (4.3.6), it suffices to evaluate

$$\int_{\mathbb{R}} \sum_{(m,n)\in\mathbb{Z}^2} \mathbb{1}_{D_{M_3,\leqslant K_3}}(\tau,m,n) \left| \int_{\mathbb{R}} \sum_{(m_1,n_1)\in\mathbb{Z}^2} u_{1,k}(\tau_1,m_1,n_1) \cdot u_{2,j}(\tau-\tau_1,m-m_1,n-n_1)\mathrm{d}\tau_1\right|^2 \mathrm{d}\tau$$

Using Cauchy-Schwarz inequality, the integral above is controlled by

$$\sup_{(\tau,m,n)\in D_{M_3,\leqslant K_3}} |A_{\tau,m,n}| \cdot ||u_{1,k}||_{L^2}^2\,||u_{2,j}||_{L^2}^2$$

where $A_{\tau,m,n}$ is defined as

$$A_{\tau,m,n} := \big\{(\tau_1,m_1,n_1)\in\mathbb{R}\times\mathbb{Z}^2, m_1\in I_k, m-m_1\in I_j, \langle \tau_1-\omega(m_1,n_1)\rangle \lesssim K_1,\ \langle \tau-\tau_1-\omega(m-m_1,n-n_1)\rangle \lesssim K_2\}$$

with the intervals

$$I_k = I_{M_1}\cap[kM_3;(k+1)M_3] \text{ and } I_j = I_{M_2}\cap[jM_3;(j+1)M_3]$$

Using the triangle inequality in $\tau_1$, we get the bound

$$|A_{\tau,m,n}| \lesssim (K_1 \wedge K_2)\,|B_{\tau,m,n}|$$

where $B_{\tau,m,n}$ is defined as

$$B_{\tau,m,n} := \big\{(m_1,n_1) \in \mathbb{Z}^2,\ m_1 \in I_k, m - m_1 \in I_j,\\ \langle \tau + \omega(m,n) - \Omega(m_1,n_1,m-m_1,n-n_1)\rangle \lesssim (K_1 \vee K_2)\}$$

and the resonant function $\Omega$ is defined as

$$\begin{aligned}\Omega(m_1,n_1,m_2,n_2) &= \omega(m_1,n_1) + \omega(m_2,n_2) - \omega(m_1+m_2,n_1+n_2)\\ &= 5m_1m_2(m_1+m_2)\alpha(m_1,m_2) - \frac{(m_1n_2 - m_2n_1)^2}{m_1m_2(m_1+m_2)}\\ &= 5m_1m_2(m_1+m_2)\alpha(m_1,m_2) - \frac{m_1m_2}{m_1+m_2}\left(\frac{n_1}{m_1} - \frac{n_2}{m_2}\right)^2 \qquad (4.3.7)\end{aligned}$$

with

$$\alpha(m_1,m_2) = m_1^2 + m_1m_2 + m_2^2 \sim M_{max}^2$$

First, in the case $K_{max} \leqslant 10^{-10}M_1M_2M_3M_{max}^2$, we estimate $|B_{\tau,m,n}|$ with the help of Lemma 4.3.2 and 4.3.3. Indeed, its projection on the $m_1$ axis is controlled by $|I_k| \wedge |I_j|$. Now, we compute

$$\left|\frac{\partial\Omega}{\partial n_1}\right| = 2\left|\frac{n_1}{m_1} - \frac{n-n_1}{m-m_1}\right| = 2\left|\frac{m}{m_1(m-m_1)}\left(5m_1(m-m_1)m\alpha(m_1,m-m_1) - \Omega\right)\right|^{1/2}$$

Thus, from the condition $|\Omega| \lesssim K_{max} \leqslant 10^{-10}M_1M_2M_3M_{max}^2$ we get

$$\left|\frac{\partial\Omega}{\partial n_1}\right| \gtrsim \left|\frac{m}{m_1(m-m_1)}\cdot m_1(m-m_1)m\alpha(m_1,m-m_1)\right|^{1/2} \sim M_3M_{max}$$

So we can estimate $|B_{\tau,m,n}|$ in this regime by

$$|B_{\tau,m,n}| \lesssim \langle |I_k| \wedge |I_j|\rangle \big\langle (K_1 \vee K_2)(M_3M_{max})^{-1}\big\rangle$$

For (4.3.4), note that we can neglect the localization $\mathbb{1}_{D_{M_3,\leqslant K_3}}$, thus we can use the argument of [ST01, Lemma 4] and assume that $m_i \geqslant 0$ on the support of $u_i$. To get a bound for $|B_{\tau,m,n}|$, we now use Lemma 4.3.4 instead of Lemma 4.3.3. Indeed, we can write

$$\begin{aligned}&\tau - \omega(m,n) - \Omega(m_1,n_1,m-m_1,n-n_1)\\ &\qquad = \tau - \omega(m,n) - 5mm_1(m-m_1)\alpha(m_1,m-m_1)\\ &\qquad\qquad + \frac{m_1^2n^2 - 2m_1mn_1n}{m_1m(m-m_1)} + \frac{m^2}{m_1m(m-m_1)}n_1^2\end{aligned}$$

which is a parabola in $n_1$ with leading coefficient

$$\left|\frac{m}{m_1(m-m_1)}\right| = \frac{1}{m_1} + \frac{1}{m-m_1} \geqslant \frac{1}{m_1 \wedge (m-m_1)}$$

Thus for a fixed $m_1$, the cardinal of the $n_1$-section is estimated by

$$\Big\langle (K_1 \vee K_2)^{1/2}(M_1 \wedge M_2)^{1/2}\Big\rangle$$

thanks to (4.3.3). So we get the final bound

$$|B_{\tau,m,n}| \lesssim \langle |I_k| \wedge |I_j| \rangle \Big\langle (K_1 \vee K_2)^{1/2} M_{min}^{1/2} \Big\rangle$$

These bounds for $|A_{\tau,m,n}|$ finally give (4.3.4) and (4.3.5) by using Cauchy-Schwarz inequality to sum over $k \in \mathbb{Z}$, since $|I_k| \lesssim M_1 \wedge M_3$ and $|I_j| \lesssim M_2 \wedge M_3$.

□

**Remark 4.3.6.** *In the context of standard Bourgain spaces, we cannot recover some derivatives in the regime $K_{max} < M_3 M_{max}$ since*

$$\Big\langle (K_1 \vee K_2)^{1/2} (M_3 M_{max})^{-1/2} \Big\rangle = 1$$

*in that case. This is the main reason for the bilinear estimate to fail in [ST01, Section 5] and for our choice of time localization on intervals of size $M_{max}^{-2}$.*

**Remark 4.3.7.** *Estimate (4.3.5) may seem rough, but a more careful analysis of the dyadic bilinear estimates in the resonant case (that is, the analogue of [GHF17, Lemma 3.1 (a)] for periodic functions) in the spirit of [Zha15, Lemma 3.1] leads to the bound*

$$(K_1 K_3)^{1/2} M_{max}^{-1} \cdot \left\{ \left( \frac{K_2}{(M_1 \wedge M_2) M_{max}} \right)^{1/2} \wedge \left[ (M_1 \wedge M_2) \left\langle \frac{K_2}{M_{min} M_{max}^3} \right\rangle \right]^{1/2} \right\}$$

*showing that, in the case $K_2 = K_{med} \leqslant M_{min} M_{max}^3$ and $M_1 \wedge M_2 = M_{min}$, (4.3.5) is actually optimal. Comparing with [GHF17, Lemma 3.1], we see why there is such a gap in regularity between the well-posedness in $\mathbb{R}^2$ and $\mathbb{T}^2$.*

As in [IKT08, Corollary 5.3], we conclude this section by summarizing the main dyadic estimates that we will use throughout the forthcoming sections.

**Corollary 4.3.8**

*Assume $M_1, M_2, M_3, K_1, K_2, K_3 \in 2^{\mathbb{N}}$ with $K_i \geqslant M_i^2$ and $f_i \in L^2(\mathbb{R} \times \mathbb{Z}^2)$ are non negative functions with the support condition $\text{supp} f_i \subset D_{M_i,K_i}$, $i = 1, 2$. Then*

$$\big|\big| \mathbb{1}_{D_{M_3,K_3}} \cdot f_1 \star f_2 \big|\big|_{L^2} \lesssim M_{min}^{1/2} M_{max}^{-1-2b} (K_{min} K_{max})^{1/2} K_{med}^b \, ||f_1||_{L^2} \, ||f_2||_{L^2} \qquad (4.3.8)$$

*for any $b \in [1/4; 1/2]$, and*

$$\big|\big| \mathbb{1}_{D_{M_3,K_3}} \cdot f_1 \star f_2 \big|\big|_{L^2} \lesssim M_1^{3/2} M_{min}^{1/2} K_{min}^{1/2} \, ||p \cdot f_1||_{L^2} \, ||f_2||_{L^2} \qquad (4.3.9)$$

**Proof :**
(4.3.8) follows directly from (4.3.4) and (4.3.5) above.

For the proof of (4.3.9), we follow [IKT08, Lemma 5.3] : we split

$$f_1 = \sum_{N \geqslant M_1^3} f_{1,N} = \mathbb{1}_{I_{\leqslant M_1^3}}(n) f_1 + \sum_{N > M_1^3} \mathbb{1}_{I_N}(n) f_1$$

such that

$$\left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot f_1 \star f_2\right|\right|_{L^2} \lesssim \sum_{N \geqslant M_1^3} N^{1/2} M_{min}^{1/2} K_{min}^{1/2} \,||f_{1,N}||_{L^2}\, ||f_2||_{L^2}$$

after using (4.3.1).
Thus, using Cauchy-Schwarz inequality in $N$, we obtain

$$\left|\left|\mathbb{1}_{D_{M_3,K_3}} \cdot f_1 \star f_2\right|\right|_{L^2} \lesssim M_{min}^{1/2} K_{min}^{1/2}\, ||f_2||_{L^2} \sum_{N \geqslant M_1^3} N^{-1/2} M_1^3\, ||p \cdot f_{1,N}||_{L^2}$$

$$\lesssim M_1^{3/2} M_{min}^{1/2} K_{min}^{1/2}\, ||p \cdot f_1||_{L^2}\, ||f_2||_{L^2}$$

□

## 4.4 Energy estimates

In this section, we prove the energy estimates which allow to control the **B**-norm of regular solutions and the $\overline{\mathbf{B}}$-norm of the difference of solutions.

**Lemma 4.4.1**

*There exists $\mu_0 > 0$ small enough such that for $T \in ]0;1]$ and $u_i \in \overline{F_{M_i}}(T)$, $i \in \{1,2,3\}$, with one of them in $F_{M_i}(T)$, then*

$$\left|\int_{[0,T]\times\mathbb{T}^2} u_1 u_2 u_3 \mathrm{d}t\mathrm{d}x\mathrm{d}y\right| \lesssim T^{\mu_0} M_{min}^{1/2} \prod_{i=1}^{3} ||u_i||_{\overline{F_{M_i}}(T)} \tag{4.4.1}$$

*If moreover $M_1 \leqslant M/16$, and $u \in \overline{F_M}(T)$, $v \in F_{M_1}(T)$, we have*

$$\left|\int_{[0,T]\times\mathbb{T}^2} P_M u \cdot P_M(P_{M_1} v \cdot \partial_x u) \mathrm{d}t\mathrm{d}x\mathrm{d}y\right|$$

$$\lesssim T^{\mu_0} M_1^{3/2} \,||P_{M_1} v||_{\overline{F_{M_1}}(T)} \sum_{M_2 \sim M} ||P_{M_2} u||^2_{\overline{F_{M_2}}(T)} \tag{4.4.2}$$

**Proof :**
From symmetry, we may assume $M_1 \leqslant M_2 \leqslant M_3$. Let $\widetilde{u_i} \in \overline{F_{M_i}}$ be extensions $u_i$ to $\mathbb{R}$, satisfying $||\widetilde{u_i}||_{F_{M_i}} \leqslant 2\, ||u_i||_{F_{M_i}(T)}$.
Let $\gamma \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that $\gamma : \mathbb{R} \to [0;1]$ with supp$\gamma \subset [-1;1]$ and satisfying

$$\forall t \in \mathbb{R}, \sum_{\nu \in \mathbb{Z}} \gamma^3(t-\nu) = 1$$

Then

$$\left|\int_{[0;T]\times\mathbb{T}^2} u_1u_2u_3 \mathrm{d}t\mathrm{d}x\mathrm{d}y\right|$$

$$= \Bigg|\sum_{|\nu|\lesssim M_{max}^2}\sum_{K_1,K_2,K_3\geqslant M_{max}^2}\int_{\mathbb{R}\times\mathbb{Z}^2}\left(\rho_{K_3}(\tau+\omega)\mathcal{F}\left\{\mathbb{1}_{[0;T]}\gamma(M_{max}^2t-\nu)\widetilde{u_3}\right\}\right)$$
$$\cdot\left(\rho_{K_1}(\tau+\omega)\mathcal{F}\left\{\mathbb{1}_{[0;T]}\gamma(M_{max}^2t-\nu)\widetilde{u_1}\right\}\right)$$
$$\star\left(\rho_{K_2}(\tau+\omega)\mathcal{F}\left\{\mathbb{1}_{[0;T]}\gamma(M_{max}^2t-\nu)\widetilde{u_2}\right\}\right)\mathrm{d}\tau\Bigg|$$

where there are at most $TM_{max}^2$ interior terms $\nu$ for which $\mathbb{1}_{[0;T]}\gamma(M_{max}^2t-\nu)=\gamma(M_{max}^2t-\nu)$, and at most 4 remaining border terms where the integral is non zero. The property of $X_M$ (4.2.8) allows us to partition the modulations at $K_i\geqslant M_{max}^2$

Let us now observe that, using (4.2.8), for the interior terms we have

$$\sup_{\nu\in\mathbb{Z}}\sum_{K_i\geqslant M_{max}^2}K_i^b\left|\left|\rho_{K_i}(\tau+\omega)\mathcal{F}\left\{\gamma(M_{max}^2t-\nu)\widetilde{u_i}\right\}\right|\right|_{L^2}\lesssim||\widetilde{u_i}||_{\overline{F_{M_i}}^b}$$

Thus, since we can take $\mu_0=1$ for those terms, (4.4.1) follows from (4.3.8) with $b=1/2$ and the estimate above.

For the remaining border terms, we use that

$$\sup_{\nu}\sup_{K_i\geqslant M_{max}^2}K_i^{1/2}\left|\left|\rho_{K_i}(\tau+\omega)\cdot\widehat{\mathbb{1}_{[0;T]}}\star\mathcal{F}\left\{\gamma(M_{max}^2t-\nu)\widetilde{u_i}\right\}\right|\right|_{L^2}\lesssim||\widetilde{u_i}||_{\overline{F_{M_i}}}$$

which follows through the same argument as for the proof of (4.2.8) (see [Rob18]). Thus we can use (4.3.8) with $b<1/2$ to get (4.4.1).

(4.4.2) then follows from the one of (4.4.1) through the same argument as in [IKT08, Lemma 6.1].

□

We can now state our global energy estimate.

**Proposition 4.4.2**

*Let $T\in]0;1[$ and $u\in\mathcal{C}([-T,T],\mathbf{E}^\infty)$ be a solution of (4.1.2) on $[-T,T]$. Then for any $\sigma\geqslant 2$,*

$$||u||_{\mathbf{B}^\sigma(T)}^2\lesssim||u_0||_{\mathbf{E}^\sigma}^2+T^{\mu_0}||u||_{\mathbf{F}(T)}||u||_{\mathbf{F}^\sigma(T)}^2 \tag{4.4.3}$$

**Proof :**

From the definition of the $\mathbf{B}^\sigma$ norm and the weight $p$, we have the first estimate

$$||u||_{\mathbf{B}^\sigma(T)}^2\lesssim\sum_{M_3>1}\sup_{t_{M_3}\in[-T;T]}\left(M_3^{2\sigma}||P_{M_3}u(t_{M_3})||_{L^2}+M_3^{2(\sigma-2)}\left|\left|\partial_x^{-1}\partial_yP_{M_3}u(t_{M_3})\right|\right|_{L^2}\right)$$

For the first term within the sum, using that $u$ is a solution to (4.1.2), we have

$$\sup_{t_{M_3}\in[-T;T]}M_3^{2\sigma}||P_{M_3}u(t_{M_3})||_{L^2}\lesssim M_3^{2\sigma}||P_{M_3}u_0||_{L^2}$$
$$+M_3^{2\sigma}\left|\int_{[0;T]\times\mathbb{T}^2}P_{M_3}u\cdot P_{M_3}(u\partial_xu)\mathrm{d}t\mathrm{d}x\mathrm{d}y\right|$$

We can divide the previous integral term into

$$\sum_{M_1\leqslant M_3/16} M_3^{2\sigma}\left|\int_{[0;T]\times\mathbb{T}^2} P_{M_3}u\cdot P_{M_3}(P_{M_1}u\cdot\partial_x u)\mathrm{d}t\mathrm{d}x\mathrm{d}y\right| + \sum_{M_1\gtrsim M_3}\sum_{M_2\geqslant 1} M_3^{2\sigma}\left|\int_{[0;T]\times\mathbb{T}^2} P_{M_3}^2 u\cdot P_{M_1}u\cdot\partial_x P_{M_2}u\mathrm{d}t\mathrm{d}x\mathrm{d}y\right|$$

Using (4.4.2) for the first one and (4.4.1) for the second one, we get the bound

$$\sum_{M_1\leqslant M_3/16} M_3^{2\sigma}M_1^{3/2}\,||P_{M_1}u||_{\overline{F_{M_1}}(T)}\sum_{M_2\sim M_3}||P_{M_2}u||^2_{\overline{F_{M_2}}(T)} + \sum_{M_1\gtrsim M_3}\sum_{M_2\geqslant 1} M_3^{2\sigma}M_2(M_2\wedge M_3)^{1/2} \cdot ||P_{M_1}u||_{\overline{F_{M_1}}(T)}\,||P_{M_2}u||_{\overline{F_{M_2}}(T)}\,||P_{M_3}u||_{\overline{F_{M_3}}(T)}$$

For the sum on the first line, we use Cauchy-Schwarz inequality to sum on $M_1$ (as we have 1/2 derivative to spare) and then sum on $M_3$ by writing $M_2 = 2^k M_3$ with $k\in\mathbb{Z}$ bounded and then a use of Cauchy-Schwarz inequality in $M_3$.
For the second line, we cut the sum into two parts $M_2\gtrsim M_3\sim M_1$ and $M_2\sim M_1\gtrsim M_3$, put $2\sigma$ derivatives on the highest frequency, and then use Cauchy-Schwarz again to sum on the lowest frequency (we have again 1/2 extra derivative) and then the biggest. Thus the term above is bounded by the right-hand side of (4.4.3).

It remains to treat the sum with the antiderivative. Proceeding similarly and writing $v := \partial_x^{-1}\partial_y u$, we get

$$\sup_{t_{M_3}\in[-T;T]} M_3^{2(\sigma-2)}\left|\left|\partial_x^{-1}\partial_y P_{M_3}u(t_{M_3})\right|\right|_{L^2}\lesssim M_3^{2(\sigma-2)}\left|\left|\partial_x^{-1}\partial_y P_{M_3}u_0\right|\right|_{L^2} + M_3^{2(\sigma-2)}\left|\int_{[0;T]\times\mathbb{T}^2} P_{M_3}v\cdot P_{M_3}(u\partial_x v)\mathrm{d}t\mathrm{d}x\mathrm{d}y\right|$$

which is analogously dominated by

$$\sum_{M_1\leqslant M_3/16} M_3^{2(\sigma-2)}M_1^{3/2}\,||P_{M_1}u||_{\overline{F_{M_1}}(T)}\sum_{M_2\sim M_3}||P_{M_2}v||^2_{\overline{F_{M_2}}(T)} + \sum_{M_1\gtrsim M_3}\sum_{M_2\geqslant 1} M_3^{2(\sigma-2)}M_2(M_2\wedge M_3)^{1/2} \cdot ||P_{M_1}u||_{\overline{F_{M_1}}(T)}\,||P_{M_2}v||_{\overline{F_{M_2}}(T)}\,||P_{M_3}v||_{\overline{F_{M_3}}(T)}$$

For the first line, we run the summation over $M_1, M_2, M_3$ as before, whereas for the second line, we split the highest frequency into $M_1^2(M_2\vee M_3)^{2(\sigma-3)}(M_2\wedge M_3)^{3/2}$ and then perform the summation as above.

□

**Remark 4.4.3.** *In the dyadic summations above, we see that we are 1/2-derivative below the energy space, thus a simple adaptation of our argument would actually yield local well-posedness in $H^{s_1,s_2}(\mathbb{T}^2)$ with $s_1 > 3/2, s_2\geqslant 0$. For our result to be more readable, we chose not to present these technical details here.*

**Remark 4.4.4.** *Even with the local well-posedness result mentioned above, our result is in sharp contrast with the local well-posedness of [GHF17] in the case of* $\mathbb{R}^2$*. This highlights the quasilinear behaviour of equation (4.1.2) in the periodic setting. From the technical point of view, the* $X_M$ *structure is used in [GHF17] on time intervals on size* $M^{-1}$*, whereas in our case, the use of the counting measure instead of the Lebesgue measure in the localized bilinear Strichartz estimates requires us to work on time intervals of size* $M^{-2}$ *which explains the gap in regularity between these results.*

To deal with the difference of solutions, we also prove the following proposition.

**Proposition 4.4.5**

*Assume* $T \in ]0;1[$ *and* $u, v \in \mathcal{C}([-T,T], \mathbf{E}^\infty)$ *are solutions to (4.1.2) on* $[-T,T]$ *with initial data* $u_0, v_0 \in \mathbf{E}^\infty$*. Then*

$$||u-v||_{\mathbf{B}(T)}^2 \lesssim ||u_0 - v_0||_{L^2}^2 + T^{\mu_0} ||u+v||_{\mathbf{F}(T)} ||u-v||_{\mathbf{F}(T)}^2 \tag{4.4.4}$$

*and*

$$||u-v||_{\mathbf{B}(T)}^2 \lesssim ||u_0 - v_0||_{\mathbf{E}}^2 + T^{\mu_0} ||v||_{\mathbf{F}^3(T)} ||u-v||_{\mathbf{F}(T)}^2 \tag{4.4.5}$$

**Proof :**
We proceed as in the previous proposition, except that now $w := u - v$ solves the equation

$$\begin{cases} \partial_t w - \partial_x^5 w - \partial_x^{-1}\partial_y^2 w + \partial_x \left( w \dfrac{u+v}{2} \right) = 0 \\ w(t=0) = u_0 - v_0 \end{cases} \tag{4.4.6}$$

For (4.4.4), we write

$$\begin{aligned} ||u-v||_{\mathbf{B}(T)}^2 = \sum_{M_3 \geqslant 1} \sup_{t_{M_3} \in \mathbb{R}} ||P_{M_3}(u-v)(t_{M_3})||_{L^2}^2 \lesssim \sum_{M_3 \geqslant 1} \Bigg\{ & ||P_{M_3}(u_0 - v_0)||_{L^2}^2 \\ & + \left| \int_{[0;T]\times\mathbb{T}^2} P_{M_3} w \cdot P_{M_3} \left( w\partial_x w + w \partial_x v + v \partial_x w \right) \mathrm{d}t\mathrm{d}x\mathrm{d}y \right| \Bigg\} \end{aligned}$$

The first integral term with $w\partial_x w$ can be estimated by $||w||_{\mathbf{F}(T)} ||w||_{\mathbf{F}}^2$ the exact same way as the first term in the previous proposition with $\sigma = 0$.

As in [Zha15], for the other two terms, we use again (4.4.1) and (4.4.2) to bound them with

$$\begin{aligned} T^{\mu_0} \Bigg\{ & \sum_{M_3 \geqslant 1} \sum_{M_1 \leqslant M_3/16} \sum_{M_2 \sim M_3} \left( M_2 M_1^{1/2} \Pi_1 + M_1^{3/2} \Pi_2 \right) \\ & + \sum_{M_3 \geqslant 1} \sum_{M_1 \gtrsim M_3} \sum_{M_2 \sim M_1} M_2 M_3^{1/2} \left( \Pi_1 + \Pi_2 \right) \\ & + \sum_{M_3 \geqslant 1} \sum_{M_1 \sim M_3} \sum_{M_2 \lesssim M_3} M_2^{3/2} \left( \Pi_1 + \Pi_2 \right) \Bigg\} \end{aligned}$$

where we have noted

$$\Pi_1 := ||w||_{F_{M_3}} ||w||_{F_{M_1}} ||v||_{F_{M_2}} \text{ and } \Pi_2 := ||w||_{F_{M_3}} ||v||_{F_{M_1}} ||w||_{F_{M_2}}$$

Observe that, as for Proposition 4.4.2 above, we have 1/2 derivative to spare. Moreover, using the relation between the $M_i$'s, we can always place all (3/2)+ derivatives on the term containing $v$, thus we can sum by using Cauchy-Schwarz to bound all these terms with the right-hand side of (4.4.4).

By the same token as for (4.4.3), we can estimate the left-hand side of (4.4.5) by $||u_0 - v_0||_{\mathbf{E}}^2$ plus two integral terms

$$\sum_{M_3 \geqslant 1} M_3^4 \left| \int_{[0;T]\times\mathbb{T}^2} P_{M_3} w \cdot P_{M_3} \left( w \partial_x w + w \partial_x v + v \partial_x w \right) \mathrm{d}t \mathrm{d}x \mathrm{d}y \right| \tag{4.4.7}$$

and

$$\sum_{M_3 \geqslant 1} \left| \int_{[0;T]\times\mathbb{T}^2} P_{M_3} W \cdot P_{M_3} \left( w \partial_x W + w \partial_x V + v \partial_x W \right) \mathrm{d}t \mathrm{d}x \mathrm{d}y \right| \tag{4.4.8}$$

where in the latter $W := \partial_x^{-1} \partial_y w$ and $V := \partial_x^{-1} \partial_y v$.

For (4.4.7), we proceed exactly as previously. Again, the first integral term has already been treated in the proof of (4.4.3). For the other terms, now we have 11/2 derivatives to distribute, and using again the relation between the $M_i$'s we can place $2^-$ derivatives on each $w$ and the remaining ones on $v$ and then run the summations, the worst case being the term $P_{M_1} w \cdot \partial_x P_{M_2} v$ in the regime $M_1 \ll M_2 \sim M_3$ since there are 5 highest derivatives, thus we need to put 3 on $v$.

It remains to treat (4.4.8). Once again, the first term within the integral appeared in the proof of the previous proposition, thus we only need to deal with the last two terms. We proceed the same way as above, since there are 3/2 derivatives to share, and $w$ can absorb 2, $V$ can absorb 1 and $v$ can absorb 3.

□

## 4.5 Short-time bilinear estimates

The aim of this section is to prove the bilinear estimates for both the equation and the difference equation. We mainly adapt [IKT08].

**Proposition 4.5.1**

*There exists $\mu_1 > 0$ small enough such that for any $T \in ]0;1]$ and $\sigma \geqslant 2$ and $u, v \in \mathbf{F}^\sigma(T)$,*

$$||\partial_x (uv)||_{\mathbf{N}^\sigma(T)} \lesssim T^{\mu_1} \left\{ ||u||_{\mathbf{F}^\sigma(T)} ||v||_{\mathbf{F}(T)} + ||u||_{\mathbf{F}(T)} ||v||_{\mathbf{F}^\sigma(T)} \right\} \tag{4.5.1}$$

**Proof :**
Using the definition of $\mathbf{F}^\sigma(T)$ (4.2.3) and $\mathbf{N}^\sigma(T)$ (4.2.4), the left-hand side of (4.5.1) is bounded by

$$\sum_{M_1, M_2, M_3} M_3^\sigma \, ||P_{M_3} \partial_x \left( P_{M_1} u \cdot P_{M_2} v \right)||_{N_{M_3}(T)}$$

For $M_1, M_2 \in 2^{\mathbb{N}}$, let us choose extensions $u_{M_1}$ and $v_{M_2}$ of $P_{M_1} u$ and $P_{M_2} v$ to $\mathbb{R}$ satisfying $||u_{M_1}||_{F_{M_1}} \leqslant 2 \, ||P_{M_1} u||_{F_{M_1}(T)}$ and similarly for $v_{M_2}$. Since the previous term is symmetrical with respect to $u$ and $v$, we can assume $M_1 \leqslant M_2$.

To treat the term above, from the definition of the $F_M^b$ and $N_M$ norms, the property of the space $X_M$ (4.2.8) and the use of Lemma 4.2.1, it suffices to show that there exists $b \in [0; 1/2)$

such that for all $K_i \geqslant M_i^2$, $i = 1, 2, 3$ and $f_i^{K_i} \in L^2(\mathbb{R} \times \mathbb{Z}^2)$ with supp$f_i^{K_i} \subset D_{M_i, \leqslant K_i}$, $i = 1, 2$ then

$$\frac{M_{max}^2}{M_3^2} \cdot M_3^{\sigma+1} \sum_{K_3 \geqslant M_3^2} K_3^{-1/2} \left|\left|\mathbb{1}_{D_{M_3, \leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2}$$
$$\lesssim M_1^2 M_2^{\sigma} (K_1 \wedge K_2)^b (K_1 \vee K_2)^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \quad (4.5.2)$$

Indeed, for a smooth partition of unity $\gamma : \mathbb{R} \to [0; 1]$ satisfying supp$\gamma \subset [-1; 1]$ and for all $t \in \mathbb{R}$

$$\sum_{\nu \in \mathbb{Z}} \gamma(t - \nu)^2 = 1$$

then define for $|\nu| \lesssim M_{max}^2 M_3^{-2}$

$$f_{1,\nu}^{K_1} := \rho_{K_1}(\tau + \omega) \cdot \mathcal{F}\left\{\gamma\left(M_{max}^2 M_3^{-2} t - \nu\right) u_{M_1}\right\}$$

with $\rho_{K_1}$ a non-homogeneous dyadic decomposition of unity partitioned at $K_1 = M_1^2$, and similarly for $f_{2,\nu}^{K_2}$. Then the norm within the sum is bounded by the left-hand side of (4.5.2) (after taking the supremum over $\nu$), wherease summing on $K_i \geqslant M_i^2$, $i = 1, 2$, using (4.2.8) and Lemma 4.2.1 and summing on $M_1, M_2$ then the right-hand side of (4.5.2) is controlled by the right-hand side of (4.5.1) (see e.g [Rob18] for the full details).

We then separate two cases depending on the relation between the $M_i$'s.

**Case A : Low × High → High.**

We assume $M_1 \lesssim M_2 \sim M_3$. In that case, for (4.5.2) it is sufficient to prove

$$M_3 \sum_{K_3 \geqslant M_3^2} K_3^{-1/2} \left|\left|\mathbb{1}_{D_{M_3, \leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2}$$
$$\lesssim \ln(M_{min}) M_{min}^{1/2} M_{max}^{-2b} (K_1 \wedge K_2)^b (K_1 \vee K_2)^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \quad (4.5.3)$$

for a $b \in [1/4; 1/2)$.

Since $K_i \geqslant M_i^2$, $i = 1, 2$, then for the large modulations, we combine (4.3.9)(for both $f_1$ and $f_2$) with the obvious bound

$$p(m_1 + m_2, n_1 + n_2) \lesssim M_1^3 M_3^{-3} p(m_1, n_1) + p(m_2, n_2) \quad (4.5.4)$$

to bound the sum for $K_3 \geqslant M_3^2 M_1^3$ by

$$M_1^{1/2} M_3^{-2b} (K_1 \wedge K_2)^{1/2} (K_1 \vee K_2)^b \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

for any $b \in [0; 1/2]$.

For the small modulations $M_3^2 \leqslant K_3 \leqslant M_3^2 M_1^3$, the sum runs over about $\ln(M_1)$ dyadic integers. Moreover, using the definition of $\Omega$ (4.3.7), we can replace (4.5.4) with

$$p(m_1 + m_2, n_1 + n_2) \lesssim p(m_2, n_2) + M_1^{1/2} M_3^{-3} K_{max}^{1/2} \quad (4.5.5)$$

Indeed, this follows from the definition of $\Omega$ which implies

$$\frac{|n|}{|m|} \lesssim \frac{|n_2|}{|m_2|} + \left(\frac{|m_1|}{|m_2 m|}|\Omega| + 5m_1^2 \alpha(m, m_2)\right)^{1/2}$$

In the case $K_{max} \leqslant 10^{-10} M_1 M_2 M_3 M_{max}^2$, we then use (4.5.5), use the bound on $K_{max}$ and then use (4.3.5) to get the estimate

$$\begin{aligned}\left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2} &\lesssim \left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot (p \cdot f_1^{K_1}) \star (p \cdot f_2^{K_2})\right|\right|_{L^2} \\ &\lesssim (K_{min} K_{max})^{1/2} K_{med}^b M_{min}^{1/2} M_{max}^{-1-2b} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}\end{aligned}$$

for any $b \in [0; 1/2]$. Thus the sum in this regime is estimated with

$$\ln(M_1) M_{min}^{1/2} M_{max}^{-2b} (K_1 \wedge K_2)^b (K_1 \vee K_2)^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

which suffices for (4.5.3).
In the regime $K_{max} \gtrsim M_1 M_2 M_3 M_{max}^2$, we apply again (4.5.5), loose a factor $K_{max}^{1/2}$ in the first term, and then use (4.3.4) instead of (4.3.5) to obtain

$$\begin{aligned}&\left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2} \\ &\qquad \lesssim K_{max}^{1/2} M_{min}^{-1/2} M_{max}^{-2} \left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot (p \cdot f_1^{K_1}) \star (p \cdot f_2^{K_2})\right|\right|_{L^2} \\ &\qquad \lesssim (K_{min} K_{max})^{1/2} K_{med}^b M_{max}^{-5/4-2b} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}\end{aligned}$$

for any $b \in [1/4; 1/2]$ which is controlled by the estimate in the previous regime.
**Case B : High $\times$ High $\to$ Low.**
We assume now $M_1 \sim M_2 \gtrsim M_3$. (4.5.3) becomes in this case

$$\begin{aligned}&M_{max}^2 M_3 \sum_{K_3 \geqslant M_3^2} K_3^{-1/2} \left|\left|\mathbb{1}_{D_{M_3,\leqslant K_3}} \cdot p \cdot f_1^{K_1} \star f_2^{K_2}\right|\right|_{L^2} \\ &\qquad \lesssim \ln(M_{max}) M_{max}^{3-2b} M_{min}^{-1/2} (K_1 \wedge K_2)^b (K_1 \vee K_2)^{1/2} \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2} \quad (4.5.6)\end{aligned}$$

For the high modulations $K_3 \geqslant M_{min}^{-2} M_{max}^6$, we use (4.3.4) along with the obvious bound

$$p(m_1 + m_2, n_1 + n_2) \lesssim \frac{M_{max}^3}{M_{min}^3} \left(p(m_1, n_1) + p(m_2, n_2)\right) \quad (4.5.7)$$

to estimate the left-hand side of (4.5.6) with

$$M_{max}^2 M_{min}^{-1/2} (K_1 \wedge K_2)^{1/2} (K_1 \vee K_2)^b \left|\left|p \cdot f_1^{K_1}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

for any $b \in [3/8; 1/2]$.

In the regime $M_3^2 \leqslant K_3 \leqslant M_3^{-2} M_2^6$ we replace (4.5.7) with

$$p(m_1 + m_2, n_1 + n_2) \lesssim M_{min}^{-2} M_{max}^2 p(m_1, n_1) + M_{min}^{-5/2} K_{max}^{1/2} \quad (4.5.8)$$

Indeed, this follows from the same argument as for (4.5.5). Proceeding then as in the previous case, we infer the final bound

$$\ln(M_{max}) M_{max}^{3-2b} M_{min}^{-1/2} (K_1 \wedge K_2)^{1/2} (K_1 \vee K_2)^b \left|\left|p \cdot f_1^{K_2}\right|\right|_{L^2} \left|\left|p \cdot f_2^{K_2}\right|\right|_{L^2}$$

□

The end of this section is devoted to the short-time bilinear estimate for the difference equation.

**Proposition 4.5.2**

*There exists $\mu_2 > 0$ small enough such that for any $T \in ]0;1]$ and $u \in \overline{\mathbf{F}}(T)$, $v \in \mathbf{F}(T)$,*

$$||\partial_x(uv)||_{\overline{\mathbf{N}}(T)} \lesssim T^{\mu_2} ||u||_{\overline{\mathbf{F}}(T)} ||v||_{\mathbf{F}(T)} \tag{4.5.9}$$

**Proof :**
Similarly to (4.5.2), now it suffices to prove

$$M_{max}^2 M_3^{-1} \sum_{K_3 \geqslant M_3^2} K_3^{-1/2} \left|\left| \mathbb{1}_{D_{M_3, \leqslant K_3}} \cdot f_1^{K_1} \star f_2^{K_2} \right|\right|_{L^2} \lesssim M_2^2 (K_1 \wedge K_2)^b (K_1 \vee K_2)^{1/2} \left|\left| f_1^{K_1} \right|\right|_{L^2} \left|\left| p \cdot f_2^{K_2} \right|\right|_{L^2} \tag{4.5.10}$$

We proceed as above, except that now $u$ and $v$ do not play a symmetric role anymore, thus we have to separate three cases. (4.5.10) then follows directly from (4.3.8) in the cases $High \times High \to Low$ and $Low \times High \to High$ and from (4.3.9) in the case $High \times Low \to High$.

□

## 4.6 Proof of Theorem 4.1.1

We finally turn to the proof of our main result.

The starting point is the local well-posedness result for smooth data of Iòrio and Nunes.

**Theorem 4.6.1 (*[IN98]*)**

*Assume $u_0 \in \mathbf{E}^\infty$. Then there exists $T = T(||u_0||_{\mathbf{E}^3}) \in ]0;1]$ and a unique solution $u \in \mathcal{C}([-T;T], \mathbf{E}^\infty)$ of (4.1.2) on $[-T;T] \times \mathbb{T}^2$.*

### 4.6.1 Global well-posedness for smooth data

In view of this result and of the conservation of the energy, for Theorem 4.1.1 (a) it remains to prove (4.1.7), which will follow from the following proposition along with (4.2.12).

**Proposition 4.6.2**

*Let $\sigma \geqslant 2$. For any $R > 0$, there exists a positive $T = T(R) \sim R^{-1/(\mu_0 \vee \mu_1)}$ such that for any $u_0 \in \mathbf{E}^\infty$ with $||u_0||_{\mathbf{E}} \leqslant R$, the corresponding solution $u \in \mathcal{C}([-T;T], \mathbf{E}^\infty)$ satisfies*

$$||u||_{\mathbf{F}^\sigma(T)} \leqslant C_\sigma ||u_0||_{\mathbf{E}^\sigma} \tag{4.6.1}$$

**Proof :**
We fix $\sigma \geqslant 2$ and $R > 0$ and take $u_0$ as in the proposition.

Let $T = T(||u_0||_{\mathbf{E}^3}) \in ]0;1]$ and $u \in \mathcal{C}([-T;T], \mathbf{E}^\infty)$ be the solution to (4.1.2) given by Theorem 4.6.1. Then, for $T' \in [0;T]$, we define

$$\mathcal{X}_\sigma(T') := ||u||_{\mathbf{B}^\sigma(T')} + ||u\partial_x u||_{\mathbf{N}^\sigma(T')} \tag{4.6.2}$$

In order to perform our continuity argument, we will use the following lemma, whose proof is a straightforward adaptation of [Rob18, Lemma 8.3].

**Lemma 4.6.3**

*Let $u \in \mathcal{C}([-T;T],\mathbf{E}^\infty)$, $\sigma \geqslant 2$ and $T \in ]0;1]$. Then $\mathcal{X}_\sigma : [0;T] \to \mathbb{R}$ defined above is continuous and nondecreasing, and furthermore*

$$\lim_{T'\to 0} \mathcal{X}_\sigma(T') \lesssim ||u_0||_{\mathbf{E}^\sigma}$$

Recalling (4.2.14)-(4.4.3)-(4.5.1) for $\sigma \geqslant 2$, we then get

$$\begin{cases} ||u||_{\mathbf{F}^\sigma(T)} \lesssim ||u||_{\mathbf{B}^\sigma(T)} + \left|\left|\partial_x(u^2)\right|\right|_{\mathbf{N}^\sigma(T)} \\ ||u||^2_{\mathbf{B}^\sigma(T)} \lesssim ||u_0||^2_{\mathbf{E}^\sigma} + T^{\mu_0} ||u||_{\mathbf{F}(T)} ||u||^2_{\mathbf{F}^\sigma(T)} \\ ||\partial_x(uv)||_{\mathbf{N}^\sigma(T)} \lesssim T^{\mu_1}\left(||u||_{\mathbf{F}^\sigma(T)} ||v||_{\mathbf{F}(T)} + ||u||_{\mathbf{F}(T)} ||v||_{\mathbf{F}^\sigma(T)}\right) \end{cases} \tag{4.6.3}$$

Thus, combining those estimates first with $\sigma = 2$, we deduce that

$$\mathcal{X}_2(T)^2 \leqslant c_1 ||u_0||^2_{\mathbf{E}} + c_2 T^{\mu_0}\mathcal{X}_2(T)^3 + c_3 T^{2\mu_1}\mathcal{X}_2(T)^4 \tag{4.6.4}$$

for $T \in ]0;1]$. Let us set $\widetilde{R} = c_1^{1/2} ||u_0||_{\mathbf{E}}$. Then we choose $T_0 = T_0(\widetilde{R}) \in ]0;1]$ small enough such that

$$c_2 T_0^{\mu_0}(2\widetilde{R}) + c_3 T_0^{2\mu_1}(2\widetilde{R})^2 < 1/2$$

Thus, using Lemma 4.6.3 above and a continuity argument, we get that

$$\mathcal{X}_2(T) \leqslant 2\widetilde{R} \text{ for } T \leqslant T_0$$

Using then (4.2.14), we deduce that

$$||u||_{\mathbf{F}(T)} \lesssim ||u_0||_{\mathbf{E}} \tag{4.6.5}$$

for $T \leqslant T_0(||u_0||_{\mathbf{E}})$.

Using again (4.2.14)-(4.4.3)-(4.5.1) for $\sigma \geqslant 3$ along with (4.6.5), we then obtain

$$\begin{cases} ||u||_{\mathbf{F}^\sigma(T)} \lesssim ||u||_{\mathbf{B}^\sigma(T)} + ||f||_{\mathbf{N}^\sigma(T)} \\ ||u||^2_{\mathbf{B}^\sigma(T)} \lesssim ||u_0||^2_{\mathbf{E}^\sigma} + T^{\mu_0} ||u_0||_{\mathbf{E}} ||u||^2_{\mathbf{F}^\sigma(T)} \\ \left|\left|\partial_x(u^2)\right|\right|_{\mathbf{N}^\sigma(T)} \lesssim T^{\mu_1} ||u_0||_{\mathbf{E}} ||u||_{\mathbf{F}^\sigma(T)} \end{cases}$$

We thus infer

$$\mathcal{X}_\sigma(T)^2 \leqslant \widetilde{c_1} ||u_0||^2_{\mathbf{E}^\sigma} + \widetilde{c_2} T^{\mu_0} R \mathcal{X}_\sigma(T)^2 + \widetilde{c_3} T^{2\mu_1} R^2 \mathcal{X}_\sigma(T)^2$$

So, up to choosing $T_0$ even smaller, such that

$$\widetilde{c_2} T_0^{\mu_0} R + \widetilde{c_3} T_0^{2\mu_1} R^2 < 1/2$$

we finally obtain (4.6.1).

□

### 4.6.2 Uniqueness

Let $u$, $v$ be two global solutions of (4.1.2) with data $u_0, v_0 \in \mathbf{E}(\mathbb{T}^2)$, and fix $T_* > 0$. Using now (4.2.15)-(4.4.4)-(4.5.9), we get that for $T \in [0; T_*]$

$$\begin{cases} ||u-v||_{\overline{\mathbf{F}}(T)} \lesssim ||u-v||_{\overline{\mathbf{B}}(T)} + \left|\left|\partial_x\left((u-v)\dfrac{u+v}{2}\right)\right|\right|_{\overline{\mathbf{N}}(T)} \\ ||u-v||^2_{\overline{\mathbf{B}}(T)} \lesssim ||u_0-v_0||^2_{L^2} + T^{\mu_0}\,||u+v||_{\mathbf{F}(T)}\,||u-v||^2_{\overline{\mathbf{F}}(T)} \\ \left|\left|\partial_x\left((u-v)\dfrac{u+v}{2}\right)\right|\right|_{\overline{\mathbf{N}}(T)} \lesssim T^{\mu_1}\,||u+v||_{\mathbf{F}(T)}\,||u-v||_{\overline{\mathbf{F}}(T)} \end{cases} \tag{4.6.6}$$

Hence we infer

$$\mathcal{X}_0(T)^2 \lesssim ||u_0-v_0||^2_{L^2} + T^{\mu_0}\,||u+v||_{\mathbf{F}(T_*)}\,\mathcal{X}_0(T)^2 + T^{2\mu_1}\,||u+v||^2_{\mathbf{F}(T_*)}\,\mathcal{X}_0(T)^2$$

where, as in the previous subsection,

$$\mathcal{X}_0(T) := ||u-v||_{\overline{\mathbf{B}}(T)} + \left|\left|\partial_x\left((u-v)\frac{u+v}{2}\right)\right|\right|_{\overline{\mathbf{N}}(T)}$$

Thus, taking $T_0 \in ]0; 1[$ small enough such that

$$T_0^{\mu_0}\,||u+v||_{\mathbf{F}(T_*)} + T_0^{2\mu_1}\,||u+v||^2_{\mathbf{F}(T_*)} < 1/2$$

we deduce that $\mathcal{X}_0(T) \lesssim ||u_0-v_0||_{L^2}$ on $[0; T_0]$ which yields $u \equiv v$ on $[-T_0; T_0]$ provided $u_0 = v_0$. Since $T_0$ only depends on $||u+v||_{\mathbf{F}(T_*)}$, we can then repeat this argument a finite number of time to reach $T_*$.

### 4.6.3 Existence

Again, in view of the conservation of mass, momentum and energy, it suffices to construct local in time solutions. To this aim we proceed as in [IKT08, Section 4].

Take $R > 0$, and let $u_0 \in \mathbf{E}$ with $||u_0||_{\mathbf{E}} \leqslant R$ and take $(u_{0,j}) \in (\mathbf{E}^\infty)^{\mathbb{N}}$ with $||u_{0,j}||_{\mathbf{E}} \leqslant R$, such that $(u_{0,j})$ converges to $u_0$ in $\mathbf{E}$. Using the same argument as for Theorem 4.1.1 (a), it suffices to prove that there exists $T = T(R) > 0$ such that $(\Phi^\infty(u_{0,j}))$ is a Cauchy sequence in $\mathcal{C}([-T; T], \mathbf{E})$. Indeed, this provides the conservation of the mass, momentum energy for the corresponding limit, which allows us to extend the result to any time $T > 0$.

Let $T = T(R)$ given by Proposition 4.6.2. For a fixed $M > 1$ and $k, j \in \mathbb{N}$, we can split

$$\begin{aligned} ||\Phi^\infty(u_{0,k}) - \Phi^\infty(u_{0,j})||_{L^\infty_T\mathbf{E}} &\leqslant ||\Phi^\infty(u_{0,k}) - \Phi^\infty(P_{\leqslant M}u_{0,k})||_{L^\infty_T\mathbf{E}} \\ &+ ||\Phi^\infty(P_{\leqslant M}u_{0,k}) - \Phi^\infty(P_{\leqslant M}u_{0,j})||_{L^\infty_T\mathbf{E}} + ||\Phi^\infty(P_{\leqslant M}u_{0,j}) - \Phi^\infty(u_{0,j})||_{L^\infty_T\mathbf{E}} \end{aligned}$$

The middle term is controlled with the standard energy estimate

$$\begin{aligned} &||u-v||^2_{L^\infty_T\mathbf{E}} \\ &\lesssim ||u_0-v_0||^2_{\mathbf{E}} + \left(||u+v||_{L^1_TL^\infty} + ||\partial_x(u+v)||_{L^1_TL^\infty}\right)||u-v||^2_{L^\infty_T\mathbf{E}} \\ &\lesssim ||u_0-v_0||^2_{\mathbf{E}} + T\,||u+v||_{L^\infty_T\mathbf{E}^{10}}\,||u-v||^2_{L^\infty_T\mathbf{E}} \end{aligned}$$

where the second line follows from a Sobolev inequality. Since

$$||\Phi^\infty(P_{\leqslant M}u_{0,j})||_{L^\infty_T\mathbf{E}^\sigma} \leqslant C_\sigma\, ||P_{\leqslant M}u_{0,j}||_{\mathbf{E}^\sigma}$$

thanks to (4.1.7), we deduce that

$$||\Phi^\infty(P_{\leqslant M}u_{0,k}) - \Phi^\infty(P_{\leqslant M}u_{0,j})||_{L^\infty_T\mathbf{E}} \leqslant C(M)\, ||u_{0,k} - u_{0,j}||_{\mathbf{E}}$$

Therefore it remains to treat the first and last terms. Writing $u := \Phi^\infty(u_{0,k})$, $v := \Phi^\infty(P_{\leqslant M}u_{0,k})$ and $w := u - v$, a use of (4.2.12) provides

$$||\Phi^\infty(u_{0,k}) - \Phi^\infty(P_{\leqslant M}u_{0,k})||_{L^\infty_T\mathbf{E}} \lesssim ||w||_{\mathbf{F}(T)}$$

As before, defining now $\widetilde{\mathcal{X}}(T') := ||w||_{\mathbf{B}(T')} + ||w\partial_x w + v\partial_x w + w\partial_x v||_{\mathbf{N}(T')}$, we get from (4.2.14)-(4.4.5)-(4.5.1) the bound

$$\widetilde{\mathcal{X}}(T')^2 \lesssim ||u_0 - v_0||^2_{\mathbb{E}} + T^{\mu_0}\, ||v||_{\mathbf{F}^3(T)}\, \widetilde{\mathcal{X}}(T')^2 + T^{2\mu_1}\, ||u+v||^2_{\mathbf{F}(T')}\, \widetilde{\mathcal{X}}(T')^2$$

From (4.6.1), we can bound

$$||v||_{\mathbf{F}^3(T)} \lesssim ||P_{\leqslant M}u_{0,k}||_{\mathbf{E}^3} \lesssim C(M)$$

and

$$||u+v||^2_{\mathbf{F}(T)} \lesssim R^2$$

Thus, taking $M$ large enough and $T < T(R)$ small enough such that

$$T^{\mu_0}C(M) + T^{2\mu_1}R^2 < 1/2$$

concludes the proof.

## 4.7 Remarks on the regularity of the flow map

Let us finally justify proposition 2.2.1.

First, note that our analysis above does not take into account that we work on the flat torus, thus the same arguments apply to a torus $\mathbb{T}^2_\lambda = \mathbb{T}\times\lambda^{-1}\mathbb{T}$ with any period $\lambda > 0$, providing a continuous flow map $\Phi_{t,\lambda} : u_0 \in \mathbb{E} = \mathbf{E}(\mathbb{T}^2_\lambda) \mapsto u(t) \in \mathbb{E}$.

As in section 3.3 in the previous chapter, we show that the flow map for equation (4.1.2) is not $\mathcal{C}^2$ by using a resonant $low-high$ interaction. It had already been used in [ST01] to prove the failure of the bilinear estimate in the Bourgain spaces.

We proceed then just as in the proof of proposition 2.1.3. Again, we set

$$u_0(x,y) := \mathcal{F}^{-1}\{L + H\}$$

with now the low and high frequency pieces being defined by

$$L(m,n) = \mathbb{1}(m=1, n=0)$$

and

$$H(m,n) = N^{-s_1-3s_2}\mathbb{1}(m = N-1, n = \alpha(N))$$

for $N \in \mathbb{N}$ to be chosen later. Note here that the frequency $\alpha(N)$ in $y$ lives in $\lambda\mathbb{Z}$. In order to annul the resonant function, we have the ansatz

$$n = N(N-1)\sqrt{5N^2 - 5N + 5}$$

For $n \in \lambda\mathbb{Z}$, writing $n = \lambda\widetilde{n}$ with $\widetilde{n} \in \mathbb{Z}$, we are thus looking for a $\lambda > 0$ such that for $N \in \mathbb{N}$ then $\sqrt{5N^2 - 5N + 5} = \lambda\widetilde{n}$. If we take $\lambda = \sqrt{5\ell}$ with $\ell \in \mathbb{N}$, setting then $X = 2N - 1$ and $Y = 2\widetilde{n}$ we are thus left with finding the integer solutions to

$$X^2 - \ell Y^2 = -3 \tag{4.7.1}$$

Note that we want $N \to +\infty$ in the following, so we have to choose $\ell \in \mathbb{N}$ such that the above hyperbola has an infinite number of integer points. Now, it is well known that the solutions $(X_k, Y_k)$ to (4.7.1) are given by

$$X_k + Y_k\sqrt{\ell} = (Y_0 + \sqrt{\ell}X_0)(u_k + \sqrt{\ell}v_k)$$

where $(X_0, Y_0)$ is a particular solution and $(u_k, v_k)$ is a solution to Pell's equation $u^2 - \ell v^2 = 1$. If $\ell$ is square free, Pell's equation has an infinite number of solutions $u_k + \sqrt{\ell}v_k = (u_0 + \sqrt{\ell}v_0)^k$ where $(u_0, v_0)$ is the fundamental solution, thus it is enough to find a square free integer $\ell$ such that (4.7.1) has at least one solution. For example, we can take $\ell = 7$ and $(X_0, Y_0) = (2, 1)$. This choice of $\lambda = \sqrt{35}$ provides an infinite set of numbers $\{N_k\}$ such that

$$\alpha(N_k) := N_k(N_k - 1)\sqrt{5N_k^2 - 5N_k + 5} \in \lambda\mathbb{Z} \tag{4.7.2}$$

With this definition of $u_0$ and for $N \in \{N_k\}$, note that we have

$$||u_0||_{H^{s_1,s_2}} \sim 1$$

With the same notations as in section 3.3, we have the lower bound for the second variation of the flow map in the direction of $\varphi$

$$||u_2(t,.)||_{H^{s_1,s_2}} \geqslant \left|\left|\mathcal{F}_{xy}^{-1} f_3(t,.)\right|\right|_{H^{s_1,s_2}}$$

with

$$\mathcal{F}_{xy}^{-1} f_3(t,x,y) = c\partial_x \left\{ \sum_{(m,n)\in\mathbb{Z}^2_\lambda} \sum_{(m_1,n_1)\in\mathbb{Z}^2_\lambda} \mathrm{e}^{i(mx+ny+t\omega(m,n))}\mathrm{e}^{it\Omega(m,n,m_1,n_1)/2} \right.$$
$$\left. t\cdot \operatorname{sinc}\left(t\Omega(m,n,m_1,n_1)/2\right) L(m_1,n_1)H(m-m_1,n-n_1)\right\}$$

From the definition of the resonant function (4.3.7) and the choice of $\alpha(N)$, we have

$$\Omega(N, \alpha(N), 1, 0) = 0$$

With the definition of $L$ and $H$, this provides the lower bound

$$|f_3(t,m,n)| \gtrsim |t|N^{1-s_1-3s_2}\mathbb{1}\left(m = N, n = \alpha(N)\right)$$

which implies

$$||u_2(t)||_{H^{s_1,s_2}} \gtrsim |t|N$$

Letting $N \to +\infty$, this raises again a contradiction with (3.3.1).

In particular, for $s_1 = 2$ and $s_2 = 0$ we have $u_0 \in \mathbf{E}^2(\mathbb{T}^2)$ with

$$||u_0||_{\mathbf{E}^2} \sim ||u_0||_{H^{2,0}} + \left|\left|\frac{n}{m}(L+H)\right|\right|_{\ell^2} \sim 1$$

and

$$||u_2(t)||_{\mathbf{E}^2} \geqslant \left|\left|\mathcal{F}_{xy}^{-1} f_3(t)\right|\right|_{H^{2,0}} \gtrsim |t|N$$

thus the same conclusion holds in $\mathbf{E}^2(\mathbb{T}^2)$.

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